# Lecture 17, November 5

**Lemma 1.** *An algebraic function $y$ is an $R$-function if and only if its monodromy group is solvable.*

*Proof.* Assume that $y$ is defined by (1) from previous lecture:

$$P_n y^n + P_{n-1} y^{n-1} + \cdots + P_0 = 0, \tag{1}$$

Let $D$ be the hypersurface

$$P_n J = 0$$

where $P_n$ is the leading coefficient and $J$ is the discriminant of (1).

Let $M$ be the monodromy group of $y$. Consider the natural homomorphism

$$p : F(D, x_0) \to M.$$

If $M$ is solvable then for some natural number $m$ the $m$-th commutator of $M$ is the identity element $e$.

The function $y$ becomes single-valued on the covering $H(D,m)$ since

$$F^m(D,x_0) \subset p^{-1}(M^m) = p^{-1}(e).$$

Conversely, if $y$ is a single-valued function on some covering $H(D,m)$ then

$$p(F^m(D,x_0)) = e.$$

But

$$p(F^m(D,x_0)) = M^m.$$

Thus the monodrogy group $M$ is solvable. □

**Theorem 2.** *If an algebraic function has unsolvable monodromy group then it cannot be represented by compositions of rational functions and radicals*

*Proof.* Lemma 11 from previous lecture and Lemma 1 show that the class of $R$-functions is closed under arithmetic operations and compositions with radicals. Lemma 1 shows that the monodromy group of any $R$-function is solvable. □

# COMPOSITIONS OF ANALYTIC FUNCTIONS AND RADICALS

Below we describe a class of multivalued functions in a domain $U \subset \mathbb{C}^N$ representable by composition of single-valued analytic functions and radicals.

**Definition 1.** *A multivalued function $y$ in $U$ is called an algebroidal function in $U$ if it satisfies an irreducible equation*

$$y^n + f_{n-1}y^{n-1} + \cdots + f_0 = 0 \tag{2}$$

*whose coefficients $f_{n-1}, \ldots, f_0$ are analytic functions in $U$.*

An algebroidal function could be considered as a continuous multivalued function in $U$ which has finitely many values.

**Theorem 3.** *A multivalued function $y$ in the domain $U$ can be represented by composition of radicals and single valued analytic functions if and only if $y$ is an algebroidal function in $U$ with solvable monodromy group.*

To prove the "only if" part one can repeat the proof of Theorem 2 replacing coverings over domains

$$\mathbb{C}^N \setminus D$$

by coverings over domains

$$U \setminus \tilde{D}$$

where $\tilde{D}$ is an analytic hypersurface in $U$.

To prove Theorem 3 in the opposite direction one can use Theorem 7 from previous lecture in the same way as it was used in the proof of Theorem 8 from previous lecture.

# LOCAL REPRESENTABILITY

Let us describe a a local version of Theorem 3.

Let $y$ be an algebroidal function in $U$ defined by (2).

One can localize the equation (2) at any point $p \in U$, i.e. one can replaced the coefficients $f_i$ of the equation (2) by their germs at $p$.

After such a localization the equation (2) can became reducible, i.e. it can became representable as a product of irreducible equations.

Thus an algebroidal functions $y$ in arbitrary small neighborhood of a point $p$ defines several algebroidal functions, which we will call *ramified germs of $y$ at $p$.*

For a ramified germ of $y$ at $p$ the monodromy group is defined (as the monodromy group of an algebroidal function in an arbitrary small neighborhood of the point $p$).

A ramified germ of an algebroidal function $y$ of one variable $x$ in a neighborhood of a point $p \in \mathbb{C}^1$ has a simple structure:

its monodromy group is a cyclic group

$$\mathbb{Z}/m\mathbb{Z}$$

and it can be represented as a composition of a radical and an analytic single-valued function:

$$y(x) = f((x-p)^{1/m})$$

where $m$ is the ramification order of $y$. The following corollary follows from Theorem 16.

**Corollary 4.** *1) If a multivalued function* $y$ *in the domain* $U$ *can be represented by composition of an algebroidal functions of one variable and single valued analytic functions*

*then the monodromy group of any ramified germ of* $y$ *is solvable.*

*2) If the monodromy group of a ramification germ of* $y$ *at* $p$ *is solvable*

*then in a small neighborhood of* $p$ *it can be represented by composition of radicals and single valued analytic functions.*

The *local monodromy group* of an algebroidal function $y$ at a point $p \in U$ is the monodromy group of the equation (2) in an arbitrary small neighborhood of the point $p$.

The ramified germs of $y$ at the point $p$ correspond to the orbits of the local monodromy group actions.

This statement can be proven in the same way as Lemma 4 from previous lecture was proved.

# APPLICATION TO THE 13-TH HILBERT PROBLEM

In 1957 A.N. Kolmogorov and V.I. Arnold proved the following totally unexpected theorem which gave a negative solution to the 13-th Hilbert problem.

**Theorem 5.** *(Kolmogorov–Arnold) Any continuous function of $n$ variables can be represented as the composition of functions of a single variable with the help of addition.*

The 13-th Hilbert problem has the following algebraic version which still remains open:

Is it possible to represent any algebraic function of $n > 1$ variables by algebraic functions of a smaller number of variables with the help of composition and arithmetic operations?

An *entire algebraic function* $y$ in $\mathbb{C}^N$ is an algebraic function defined in $U = \mathbb{C}^N$ by an equation (2) whose coefficient $f_i$ are polynomials. An entire algebraic function could be considered as a continuous algebraic function.

It turns out that in Kolmogorov–Arnold Theorem one cannot replace continuous functions by entire algebraic functions.

**Theorem 6.** *If an entire algebraic function can be represented as a composition of polynomials and entire algebraic functions of one variable, then its local monodromy group at each point is solvable.*

*Proof.* Theorem 6 follows from from Corollary 4. □

**Corollary 7.** *A function $y(a,b)$, defined by equation*

$$y^5 + ay + b = 0,$$

*cannot be expressed in terms of entire algebraic functions of a single variable by means of composition, addition and multiplication.*

*Proof.* Indeed, it is easy to check that the local monodromy group of $y$ at the origin is the unsolvable permutation group $S_5$. □

Division is not a continuous operation and it destroys the locality. One cannot add division to the operations used in Theorem 6.

It is easy to see that the function $y(a, b)$ from Corollary 4 can be expressed
in terms of entire algebraic functions of a single variable by means of composition and arithmetic operations:

$$y(a, b) = g(b/\sqrt[4]{a^5})\sqrt[4]{a},$$

where $g(u)$ is defined by equation

$$g^5 + g + u = 0.$$

The following particular case of the algebraic version of the 13-th Hilbert problem still remains open.

**OPEN PROBLEM** Show that there is an algebraic function of two variables which cannot be expressed in terms of algebraic functions of a single variable by means of composition and arithmetic operations.