Lecture 5, September 24

LAGRANGE INTERPOLATION POLYNOMIALS

WITH MULTIPLE INTERPOLATION POINTS

Consider a set $\Lambda \subset K$ containing k points

$$\lambda_1, \dots, \lambda_k$$

equipped with natural multiplicities

$$m_1 + \dots + m_k$$

whose sum equals to n.

The monic polynomial

$$T(t) = (t - \lambda_1)^{m_1} \cdot \dots \cdot (t - \lambda_k)^{m_k}$$

is naturally related to the set Λ.

Definition 1. *The polynomial*

$$L(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$$

of degree $< n$ *is called the Lagrange interpolating polynomial with the interpolation points* $\lambda_1, \ldots, \lambda_k$ *of multiplicities* $m_1, \ldots, m_k$ *and the interpolation data*

$$c_1^{(0)}, \ldots c_1^{(m_1-1)}, c_2^{(0)}, \ldots, c_2^{(m_2)-1}, c_k^{(0)}, \ldots, c_k^{(m_k-1)},$$

if for any i, m *where* $1 \leq i \leq k, 0 \leq m < m_i$ *the following identities hold:*

$$L^{(m)}(\lambda_i) = c_i^{(m)}$$

where $L^{(0)}(\lambda_i) = L(\lambda_i)$ *and for* $m > 0$ $L^{(m)}(\lambda_i)$ *is the value of* m*-th derivative of* L *at the point* λ_i*.*

Theorem 1. *There exists and unique Lagrange interpolation polynomial for any interpolation points* $\lambda_1, \ldots, \lambda_k \in K$ *of multiplicities* $m_1, \ldots, m_k$ *such that* $\sum m_i = n$ *for any set of interpolation data*

$$c_1^{(0)}, \ldots c_1^{(m_1-1)}, c_2^{(0)}, \ldots, c_2^{(m_2)-1}, c_k^{(0)}, \ldots, c_k^{(m_k-1)}.$$

Proof. Consider

$$L(t) = a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}$$

as a polynomial with undetermined coefficients

$$a_0, \ldots, a_{n-1}.$$

Each identity

$$L^{(m)}(\lambda_i) = c_i^{(m)}$$

provides a linear equation on the coefficients.

Thus we have n linear equations for n unknowns.

The corresponding homogeneous system has only zero solution. Indeed its solution corresponds to the polynomial L of degree $< n$ having roots

$$\lambda_1, \dots, \lambda_k$$

with multiplicities

$$m_1, \dots, m_k$$

whose sum is n. But if a polynomial of degree $< n$ has n roots counting with multiplicities then it identically equal to zero.

Because the homogeneous system has only zero solution, any corresponding non homogeneous system has exactly one solution. □

Let $\lambda_i \in \Lambda$ be an interpolation point of multiplicity m_i

Definition 2. *A principal Lagrange resolvent* $\widehat{T}_i$ *corresponding to* λ_i *is the Lagrange interpolation polynomial with the following interpolation data:*

1) $c_j^{(m)} = 0$ *for all pairs* (j, m) *with* $1 \le j \le k, 0 \le mm \le< m_i$*, except the pair* $(i, 1)$*,*

2) $c_i^{(1)} = 1$*.*

Definition 3. *Let* (T) *be the principal ideal in the polynomial ring* $K[t]$ *consisting of all polynomials divisible by the polynomial*

$$T(t) = (t - \lambda_1)^{m_1} \cdot \dots \cdot ((t - \lambda_k)^{m_k}.$$

We will write

$Q_1 \equiv Q_2 \mod (T)$ *if the polynomial* $Q_1(t) - Q_2(t)$ *is divisible by the polynomial* $T(t)$*.*

Proposition 1. *The following statements hold.*

Principal Lagrange resolvents $\widehat{T}_i$ *satisfy the following relations:*
$\widehat{T}_1 + \cdots + \widehat{T}_k - 1 \equiv 0 \mod (T)$. *Moreover*
$\widehat{T}_1 + \cdots + \widehat{T}_k - 1 = 0.$
$\widehat{T}_i\widehat{T}_j \equiv 0 \mod (T)$ *for* $i \neq j$,
$\widehat{T}_i^2 \equiv \widehat{T}_i, \mod (T)$,
$(t - \lambda_i)^{m_i}\widehat{T}_i \equiv 0 \mod (T)$.

Proof. Let us show for example that $\widehat{T}_i^2 \equiv \widehat{T}_i \mod (T)$. Polynomial $\widehat{T}_i^2$ has the same interpolation data as the polynomial $\widehat{T}_i$. Indeed since about the point λ_i the polynomial $\widehat{T}_i$ equal to $1 + o(t - \lambda_i)^{m_i}$. About any other point $\lambda_j \neq \lambda_i$ the polynomial $\widehat{T}_i$ equal to $0 + o(t - \lambda_j)^{m_j}$. Thus $\widehat{T}_i^2$ equal to $1 + o(t - \lambda_i)^{2m_i}$ about λ_i and equal to $0 + (t - \lambda_j)^{2m_j}$ about λ_j. So the difference of these polynomials is divisible by T.

All other identities modulo (T) from Proposition have the similar proof.

The identity $\widehat{T}_1 + \cdots + \widehat{T}_k - 1 \equiv 0 \mod (T)$ implies the identity $\widehat{T}_1 + \cdots + \widehat{T}_k - 1 = 0$ since the degree of the polynomial $\widehat{T}_1 + \cdots + \widehat{T}_k - 1$ is smaller than the degree of the polynomial T.

$\square$

Consider a vector space V (possibly, infinite dimensional) over the field K and a linear operator

$$A : V \to V.$$

Suppose that the operator A satisfies a polynomial equation

$$T(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n E = 0,$$

where $a_i \in K$, and E is the identity operator.

Assume that the polynomial

$$T(t) = t^n + a_1 t^{n-1} + \cdots + a_n$$

has k different roots

$$\lambda_1, \ldots, \lambda_k$$

in the field K of multiplicities $m_1, \ldots, m_k$.

Definition 4. *The operator*

$$\widehat{L}_i = \widehat{T}_i(A),$$

where $\widehat{T}_i(t)$ *is the principal Lagrange resolvent corresponding to* λ_i *will be called the* principal Lagrange resolvent *of the operator* A *corresponding to the root* λ_i*.*

Definition 5. *For every vector* $x \in V$*, the vector* $\widehat{x}_i = \widehat{L}_i x$ *will be called the principal Lagrange resolvent (corresponding to the root* λ_i*) of the operator.*

Problem 1. *Prove Proposition stated below.*

Proposition 2. *The following statements hold.*

1. *Principal Lagrange resolvents L_i of the operator A satisfy the following relations:*

   $L_1 + \cdots + L_k = E$,

   $L_i L_j = 0$ *for* $i \neq j$,

   $L_i^2 = L_i$,

   $(A - \lambda_i E)^{m_i} L_i = 0$.

2. *Every vector $x \in V$ is representable as the sum of its principal Lagrange resolvents, i.e.* $x = \widehat{x}_1 + \cdots + \widehat{x}_k$.

   Moreover, all nonzero principal Lagrange resolvents $widehat x_i$ *of the vector x are linearly independent and are equal to generalized eigenvectors of multiplicity m_i of the operator A with the corresponding eigenvalues λ_i, i.e.*

$$A - \lambda_i E)^{m_i} \widehat{x}_i = 0.$$

Let $U \subset K$ be a set in the ground field K containing a set of interpolation points Λ.

Let $y : U \to K$ be any function such that its derivatives $y^{(m)}(\lambda_i)$ of the order $m < m_i$ at any interpolation point λ_i of multiplicity m_i are defined.

This condition is satisfied for any rational function y whose poles do not belong to the set Λ.

If $K = \mathbb{C}$ this condition is also satisfied for a meromorphic function y while poles do not belong to Λ.

Definition 6. *An interpolation polynomial of y with the interpolations points*

$$x_1, \dots, x_k$$

of multiplicities

$$m_1, \dots, m_k$$

$m_1 + \cdots + m_k = n$ *is a polynomial L which is equal to y about the points*

$$x_1, \dots, x_k$$

up to the orders $m_1 - 1, \dots, m_k - 1$, i.e. the following identities hold:

$$L(x_i) = y(x_i), \dots, L^{(m_i-1)}(x_i) = y^{(m_i-1)}(x_i)$$

for all $1 \leq i \leq k$.

The unique interpolation polynomial whose degree is smaller than

$n = m_i + \cdots + m_k$ *is called the Lagrange interpolation polynomial of* y.

Example 1. *The interpolation polynomial* L *of a function* $y(x)$ *with one interpolation point*

$$x_1$$

of multiplicity

$$m$$

coincides with the degree $m-1$ *Taylor polynomial of* y *at the point* x_1 *t.i.* $L(x)$ *is the following function:*

$$L(x) =$$

$$= y(x_1) + y'(x_1)(x - x_1) + \cdots + \frac{1}{(m-1)!} y^{(m-1)}(x_1)(x - x_1)^{m-1}.$$

Problem 2. *Prove the Proposition and the Theorem stated below.*

Proposition 3. *Assume that a degree n polynomial T has k different roots $x_1, \ldots, x_k \in K$ of multiplicities $m_1, \ldots, m_k$ with $\sum m_i = n$.*

Then for any polynomial Q a polynomial L is its interpolation polynomial with the interpolation points equal to the roots of T (with the corresponding multiplicities) if and only if

$$Q - L \equiv 0 \mod (T).$$

The Lagrange interpolation polynomial of Q with the above interpolation points is the remainder R of its division of Q by T

Theorem 2. *Let A be a $(n \times n)$-matrix with entries in the field K having k different eigenvalues $x_1, \ldots, x_k$ and let Q be any polynomial over K. Then*

$$Q(A) = R(A),$$

where R is the Lagrange interpolation polynomial of Q with the interpolation points $x_1, \ldots, x_k$.

FACTOR RING $K[t]/(T)$ AND INTERPOLATION

Let $\Lambda = \{\lambda_i\} \subset K$ be set of k points equipped with multiplicities $m_1, \ldots, m_k$ with $\sum m_i = n$ be the set of the root of a degree n polynomial T.

The interpolation of polynomials with the interpolation points Λ is very related to the ring

$$K[t]/(T).$$

Indeed a polynomial L is an interpolation polynomial of a polynomial Q if

$$L \equiv Q \mod T$$

and the Lagrange interpolation polynomial of Q is its interpolation polynomial of the smallest degree.

Let me recall the following Lemma

Lemma 3. *A polynomial Q is invertible if $K[t]/(T)$ if and only if q have no common divisors with t, i.e. if the greater common divisor of q an t is 1.*

Proof. Indeed if I and Q have a divisor D of positive degree then RQ for any R has divisor D and can not be $\equiv 1 \mod (T)$.

It qcd of Q and t is 1, then one can find L_1 and L_2 such that

$$L_1 Q + L_2 T \equiv 1 \mod (T).$$

(one can fined L_1 and L_2 constructively using Euclidian algorithm.)The polynomial $L - 1$ is the inverse element to $Q1$ modulo (T). □

Assume that $\Lambda_1 \subset K$ and $\Lambda_2 \subset K$ are not intersecting
finite sets of points equipped with multiplicities $m_1, \ldots, m-p$ and $l_1, \ldots, l_q$ with $\sum m_i = n_1$ and $\sum l_i = n_2$. Let T_1, T_2 be monic polynomials of degrees n_1, n_2 whose sets of roots are Λ_1 and Λ_2

The interpolation problem with the interpolation set

$$\Lambda = \Lambda_1 \cup \Lambda_2$$

can be reduced to interpolations with the interpolation sets Λ_1 and λ_2

Since the polynomials T_1 and T_2 have no common roots the polynomial T_1 can be invert in the ring $K[t]/(T_2)$ and the polynomial T_2 can be invert in $K[t]'(T_1)$.

Problem 3. *The Lagrange interpolation polynomial $L(Q)$ of a polynomial Q with the interpolation set Λ is equal to*

$$[QT_2^{-1}]_1 T_2 + [QT_1^{-1}]_2 T_1.$$

Where $[QT_2^{-1}]_1$ is the Lagrange interpolation polynomial of the rational function QT_2^{-1} with the interpolation set Λ_1 and $[QT_1^{-1}]_2$ is the Lagrange interpolation polynomial of the rational function QT_1^{-1} with the interpolation set Λ_2

could be reduced to the problems o
of interpolation points $\Lambda = \Lambda_1 \cup \Lambda_2$.
Assume that an operator A satisfies a polynomial equation

$$T(A) = 0.$$

Where T is a degree n polynomials over K whose roots Λ belong to K.

Problem 4. *1) Assume that* $0 \notin \Lambda$ *and* $L(t)$ *is an interpolation polynomial with the interpolation set* Λ *of the function*

$$\frac{1}{t}$$

over the set Λ.

Then the operator A *is invertible an*

$A^{-1} = L(A)$.

2) Consider a rational function $f(t) = P(t)/Q(t)$ *such that* Q *does not vanish at any point of* Λ. *Let* L *be an interpolation polynomial of* f *with the interpolation set* Λ. *Then the operator* $P(A)[Q(A)]^{-1}$ *is defined and it is equal to* $L(A)$.

3) Assume that $F(t)$ *is an entire function of complex variable* t, *and let its interpolation polynomial with the interpolation set* λ. *Then* $F(A) = L(A)$.

4) Assume that the polynomial T *is equal to* $(t-\lambda_1)^n$ *and* Λ *is the* *set containing one point* λ_1 *of multiplicity* n. *Let* $F(t)$ *be a rational function such that* λ_1 *is not a pole of* f, *or assume that* $F(t)$ *is an entire function of complex variable* t. *Then* $F(A) = T_{\lambda_1}^{(n-1)}(A)$ *where* $T_{\lambda_1}^{(n-1)}(t)$ *is the degree* $n-1$ *Taylor polynomial of* F *at the point* (λ_1). *Thus*

$$F(A) == T_{\lambda_1}^{(n-1)}(A) =$$

$$= F(\lambda_1)E + F'(\lambda_1)(A-E) + \cdots + \frac{1}{(n-1)!}F^{(n-1)}(\lambda_1)(A-E)^{n-1}.$$

Combining the previous problem with last problem one obtains the following result.

Theorem 4. *Let $L_i(A)$ be the principal resolvent of an operator A corresponding to a root λ_i of a polynomial T such that $T(A) = 0$. Let F be a rational function in t whose poles do not contain any of the roots λ_i of T. Then*

$$F(A) = \sum_i \left(T_{\lambda_i}^{(m_i-1)} F \right) L_i(A),$$

where $\left(T_{\lambda_i}^{(m_i-1)} F \right)$ is the Taylor polynomial of degree $m_i - 1$ of F at the point λ_i.

Let us present a formula for the principal resolvent $\widehat{L}_i$.

Problem 5. *Let* $\Lambda = \{\lambda_i\} \subset K$ *be a set containing* k *point equipped with multiplicities* m_i *with* $\sum \lambda_i = n$ *and let* T *be the monic degree* n *polynomial whose set of roots is* Λ. *The the principal resolvent* $\widehat{L}_i$ *of* T *corresponding to* λ_i *is given by the following formula*

$$\left(T_{\lambda_i}^{(m_i-1)} F\right) \prod_{j \neq i} (t - \lambda_j),$$

where

$$F = \frac{1}{\prod_{j \neq i} (t - \lambda_j)}.$$

where $\left(T_{\lambda_i}^{(m_i-1)} F\right)$ *is the Taylor polynomial of degree* $m_i - 1$ *of* F *at the point* λ_i.