Lecture 24, December 8

NECESSARY CONDITIONS OF REPRESENTABILITY BY QUADRATURES, k-QUADRATURES AND GENERALIZED QUADRATURES

The main theorem, together with the computation of classes of group pairs, provides topological obstructions to representability of functions by generalized quadratures, by k-quadratures and by quadratures. Let us summaries this information.

Let us start with the definition of the class of functions representable by single valued $\mathcal{S}$-functions and quadratures (k-quadratures, generalized quadratures). We will define these classes by providing a list of basic functions and a list of admissible operations.

FUNCTIONS REPRESENTABLE BY SINGLE VALUED $\mathcal{S}$-FUNCTIONS AND QUADRATURES

The *list of basic functions*: single valued $\mathcal{S}$-functions.

The *list of admissible operations*: compositions, meromorphic operations, differentiation, integration.

FUNCTIONS REPRESENTABLE BY SINGLE VALUED $\mathcal{S}$-FUNCTIONS AND k-QUADRATURES

This class of functions is defined in the same way. We only need to add the solutions of degree $\leq k$ algebraic equations to the list of admissible operations.

FUNCTIONS REPRESENTABLE BY SINGLE VALUED $\mathcal{S}$-FUNCTIONS AND GENERALIZED QUADRATURES

This class of functions is defined in the same way. We only need to add the operation of solving algebraic equations to the list of admissible operations.

It is readily seen from the definition that the class of functions representable by single valued $\mathcal{S}$-functions and quadratures (k-quadratures, generalized quadratures) contains the class of functions representable by quadratures (k-quadratures, generalized quadratures).

It is clear that the classes of functions just defined are much broader than their classical analogs. Therefore, say, the claim that a function f does not belong to the class of functions representable by single valued $\mathcal{S}$-functions and quadratures is considerably stronger than the claim that f is not representable by quadratures.

Proposition 1. *The class of functions representable by single valued $\mathcal{S}$-functions and quadratures (k-quadratures, generalized quadratures) is contained in the class of $\mathcal{S}$-functions.*

This proposition follows immediately from the theorem on the stability of the class of $\mathcal{S}$-functions.

A RESULT ON GENERALIZED QUADRATURES. *The closed monodromy pair $[f]$ of a function f representable by generalized quadratures has a normal tower such that each divisor of this tower is either a finite group or a commutative group. Furthermore, this condition is fulfilled for the closed monodromy pair $[f]$ of every function f representable by single valued $\mathcal{S}$-functions and generalized quadratures. If the function f is almost normal,* *then the monodromy group of the function $[f]$ also satisfies this condition.*

A RESULT ON K-QUADRATURES. *The closed monodromy pair $[f]$ of a function f representable by k-quadratures, has a normal tower such that each divisor of this tower is either a subgroup of the group $S(k)$, or a commutative group. Furthermore, this condition is fulfilled for the closed monodromy pair $[f]$ of every function f representable by single valued $\mathcal{S}$-functions and k-quadratures. If the function f is almost normal, then the monodromy group of the function f also satisfies this condition.*

A RESULT ON QUADRATURES. *The closed monodromy group of a function f representable by quadratures is solvable. Furthermore, the closed monodromy group of every function f representable by single valued $\mathcal{S}$-functions and quadratures, is solvable.*

To prove these results, it suffices to apply the main theorem to the classes

$$\hat{\mathcal{M}}\langle \mathbb{C}, \mathcal{K}\rangle, \hat{\mathcal{M}}\langle \mathbb{C}, S(k)\rangle \quad \text{and} \quad \hat{\mathcal{M}}\langle \mathbb{C}\rangle$$

of $\mathcal{S}$-functions, and to use the computation of the classes

$$\mathcal{M}\langle \mathbb{C}, \mathcal{K}\rangle, \mathcal{M}\langle \mathbb{C}, S(k)\rangle \quad \text{and} \quad \mathcal{M}\langle \mathbb{C}\rangle.$$

Let us now give examples of functions not representable by generalized quadratures. Let the Riemann surface of a function f be the universal covering of the region $S^2 \setminus A$, where S^2 is the Riemann sphere, and A is a finite set containing at least three points. Then *the function f is not representable by single-valued $\mathcal{S}$-functions and generalized quadratures.*

Indeed, the function f is an almost normal function. The closed monodromy group of the function f is free and noncommutative, since the fundamental group of the region $S^2 \setminus A$ is free and noncommutative.

EXAMPLE 1. Consider the function f that maps the upper half-plane conformally onto a triangle with zero angles, bounded by arcs of circles. The function f is the inverse of the Picard modular function. The Riemann surface of the function f is the universal covering of the sphere with 3 punctures, hence the function f is not representable by single valued $\mathcal{S}$-functions and generalized quadratures.

Note that the function f is closely related to the elliptic integrals

$$K_1(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad \text{and}$$

$$K_2(k) = \int_0^{\frac{1}{k}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Each of the three functions K_1, K_2 and f can be expressed by quadratures through each function. Hence *each of the integrals K_1 and K_2 is not representable by $\mathcal{S}$-functions and generalized quadratures.*

Example 1 admits a substantial generalization. Below we provide the list of all polygons bounded by arcs of circles that are the images of the upper half-plane under functions representable by generalized quadratures.

Example 2. Let f be a k-valued algebraic function with simple ramification points located in different geometric points of the Riemann sphere. *For $k > 4$, the function f is not representable by single valued $\mathcal{S}$-functions and $(k-1)$-quadratures, compositions and meromorphic operations. In particular, the function f is not representable by $(k-1)$-quadratures.*

Indeed, a loop around a simple ramification point of the function f gives rise to a transposition of the set of branches of this function. The monodromy group of the function f is a transitive permutation group generated by transpositions, i.e. the group $S(k)$.

For $k > 4$, the group $S(k)$ does not lie in the class

$$\mathcal{M}\langle \mathbb{C}, S(k-1)\rangle.$$

In fact, the topological results on non-representability of functions by quadratures (k-quadratures and generalized quadratures) can be

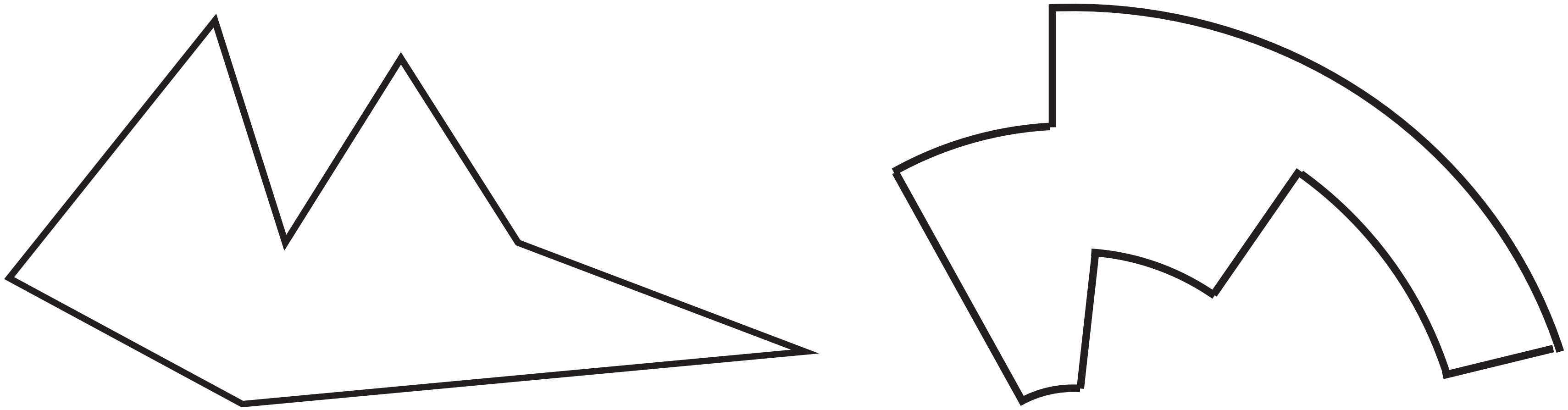

Figure 1: The first and the second cases of integrability

generalized to the case of functions of several complex variables (you san find this result in my book).

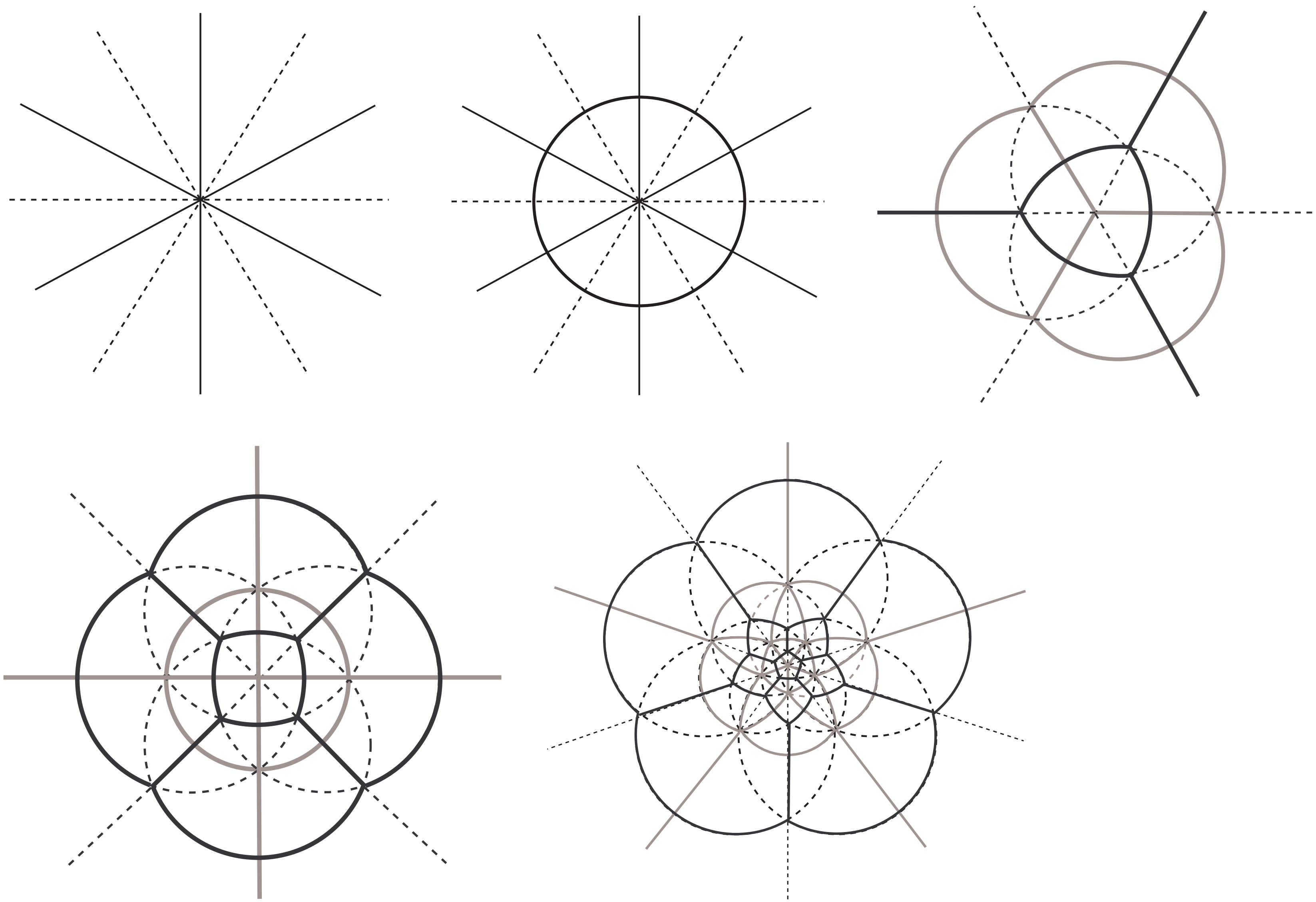

Figure 2: Finite nets of great circles

CLASSES OF SINGULAR SETS AND A GENERALIZATION OF THE MAIN THEOREM

Above we considered $\mathcal{S}$-functions, i.e. multivalued analytic functions of a complex variable, whose singular sets are at most countable. Let $\mathcal{S}$ be the class of all at most countable subsets of the Riemann sphere S^2. Let us list the properties of the class $\mathcal{S}$ that we have actively used:

1. If $A \in \mathcal{S}$, then the set $S^2 \setminus A$ is dense and locally path connected,
2. There exists a nonempty set A such that $A \in \mathcal{S}$,
3. If $A \in \mathcal{S}$ and $B \subseteq A$, then $B \in \mathcal{S}$,
4. If $A_i \in \mathcal{S}$, $i = 1, 2, \dots$, then $\bigcup_1^\infty A_i \in \mathcal{S}$,
5. Let U_1 and U_2 be open subsets of the sphere and $f : U_1 \to U_2$ an invertible analytic map, then if $A \subseteq U_1$ and $A \in \mathcal{S}$, then $f(A) \in \mathcal{S}$.

A *complete class of sets* is any class of subsets of the Riemann sphere satisfying properties 1–5. A multivalued analytic function will be called a *Q-function* if its singular point set lies in some complete class Q of sets.

All definitions and theorems carry over to Q-functions. Thus, for example, we have

A version of the main theorem. *For every complete class Q of sets and every complete class $\mathcal{M}$ of group pairs, the class $\hat{\mathcal{M}}$ consisting of all Q-functions f such that $[f] \in \mathcal{M}$ is stable under differentiation, composition and meromorphic operations. If, in addition,*

1. $\mathbb{C} \in \mathcal{M}$, *then the class $\hat{\mathcal{M}}$ of Q-functions is stable under the integration,*
2. $S(k) \in \mathcal{M}$, *then the class $\hat{\mathcal{M}}$ of Q-functions is stable under the operation of solving algebraic equations of degrees at most k.*

Let us give an example of a complete class of sets. Let X_α be the set of all subsets of the Riemann sphere with the zero Hausdorff measure of weight α. It is not hard to show that for $\alpha \leq 1$ the set X_α is a complete class of subsets of the sphere. Note that the new formulation of the main theorem allows us to strengthen all negative results. Consider, for example, the result of non-representability of functions by quadratures. (The results of non-representability by k-quadratures and by generalized quadratures can be generalized in the same way.) Define the following class of functions.

FUNCTIONS REPRESENTABLE BY SINGLE VALUED X_1-FUNCTIONS AND QUADRATURES.

The list of BASIC FUNCTIONS: single valued X_1-functions.

The LIST OF ADMISSIBLE OPERATIONS: composition, meromorphic operations, differentiation, integration. *By the new formula-*

tion of the main theorem, a $\mathcal{S}$-function having an unsolvable monodromy group is not only non-representable by quadratures, but also non-representable by single valued X_1-functions and quadratures.

Corollary 1. *If a polygon G bounded by circle arcs satisfies neither of the three integrability conditions, then the function f_G cannot be expressed by generalized quadratures, compositions and meromorphic operations through single valued X_1-functions.*