Lecture 22, December 1

MONODROMY GROUP

Let us discuss different notions related to the monodromy group.

MONODROMY GROUP WITH A FORBIDDEN SET

The monodromy group of a $\mathcal{S}$-function f with a forbidden set A is the group of all permutations of branches of f that correspond to loops around the points of A. We now give a precise definition.

Let F_a be the set of all germs of a $\mathcal{S}$-function f at a point a not lying in some forbidden set A. Take a closed path γ in $S^2 \setminus A$ that originates at the point a. The continuation of every germ in the set F_a along the path γ is another germ in the set F_a.

Thus, to every path γ, we assign a map of the set F_a to itself; moreover, homotopic paths in $S^2 \setminus A$ give rise to the same map. The composition of paths gives rise to the product of maps. This defines a homomorphism τ from the fundamental group of the set $S^2 \setminus A$ to the group $S(F_a)$ of invertible transformations of the set F_a. This homomorphism will be called the homomorphism of *A-monodromy*.

Definition 1. *The monodromy group of a $\mathcal{S}$-function f with a forbidden set A or, for short, the* group of A-monodromy *is, by definition, the image of the fundamental group $\pi_1(S^2 \setminus A, a)$ in the group $S(F_a)$ under the homomorphism τ.*

Proposition 1. *The following properties hold.*

1. *The A-monodromy group of a $\mathcal{S}$-function is independent of the choice of the point a.*

2. *The A-monodromy group of a $\mathcal{S}$-function f acts transitively on the branches of f.*

Both claims can be easily proved with the help of Lemma on removing of a curve from a countable set. Let us give, say, a proof of the second claim.

Proof. Let $f_{1,a}$ and $f_{2,a}$ be any two germs of the function f at the point a.

Since the germs $f_{1,a}$ and $f_{2,a}$ are equivalent, there exists a path γ such that, under continuation along this path, the germ $f_{1,a}$ is transformed into the germ $f_{2,a}$.

By Lemma there exists an arbitrarily close path $\hat{\gamma}$ avoiding the set A. If the path $\hat{\gamma}$ is sufficiently close to the path γ, then the corresponding permutation of branches takes the germ $f_{1,a}$ to the germ $f_{2,a}$.

(It is enough to take $\hat{\gamma}$ such that

$$|\hat{\gamma}(t) - \gamma(t)| < R_0$$

where R_0 is the smallest value of the radius of convergence of power series obtained from $f_{1,a}$ by analytic continuation along γ.) $\square$

CLOSED MONODROMY GROUP

The dependence of the A-monodromy group on the choice of the set A suggests to introduce something like

the Tychonoff topology (i.e. the direct product topology) on the permutation group of the branches.

It turns out that the closure of the A-monodromy group with respect to this topology is already independent of the set A.

Definition 2 (Topology on the group $S(M)$)**.** *The group $S(M)$ of invertible transformations of the set M is equipped with the following topology. For every finite set $L \subset M$, define a neighborhood U_L of the identity as the collection of transformations $p \in S(M)$ such that $p(l) = l$ for $l \in L$.*

The neighborhoods of the form U_L, where L runs over all finite subsets of M, form a basis of neighborhoods of the identity.

Lemma 1 (on the closure of the monodromy group)**.** *The closure of the monodromy group of a $\mathcal{S}$-function f with a forbidden set A in the group $S(F)$ of all permutations of the branches of f is independent of the choice of a forbidden set A.*

Proof. Let A_1 and A_2 be two forbidden sets of the function f, and F_a the collection of branches of f at a point a, $a \notin A_1 \cup A_2$.

Let $\Gamma_1, \Gamma_2 \subseteq S(F_a)$ be the monodromy groups of f with these forbidden sets. It suffices to show that, for every permutation $\mu_1 \in \Gamma_1$ and for every finite set $L \subseteq F_a$, there exists a permutation $\mu_2 \in \Gamma_2$ such that $\mu_1|_L = \mu_2|_L$.

Suppose that a path

$$\gamma \in \pi_1(S^2 \setminus A_1, a)$$

gives rise to the permutation μ_1.

Since the set L is finite, any path

$$\hat{\gamma} \in \pi_1(S^2 \setminus A_1, a)$$

sufficiently close to the path γ gives rise to a permutation $\hat{\mu}_1$ that coincides with μ_1 on the set L, $\mu_1|_L = \hat{\mu}_1|_L$. By the lemma, such a path $\hat{\gamma}$ can be chosen so that it does not intersect the set A_2.

In this case, the permutation $\hat{\mu}_1$ lies in the group Γ_2. $\square$ $\square$

The lemma justifies the following definition:

Definition 3. The closed monodromy group *of a $\mathcal{S}$-function f is the closure in the group $S(F)$ of the monodromy group of the function with any forbidden set A.*

TRANSITIVE ACTION OF A GROUP ON A SET AND THE MONODROMY PAIR OF A $\mathcal{S}$-FUNCTION

The monodromy group of a function f is not only an abstract group, but also a transitive permutation group of the branches of this function.

Let us recall an algebraic description of transitive group actions.

An *action* of a group Γ on a set M is a homomorphism τ of the group Γ to the group $S(M)$.

Two actions $\tau_1 : \Gamma \to S(M_1)$ and $\tau_2 : \Gamma \to S(M_2)$ are said to be *equivalent* if there exists a one-to-one correspondence $q : M_1 \to M_2$ such that $\overline{q} \circ \tau_1 = \tau_2$, where $\overline{q} : S(M_1) \to S(M_2)$ is the isomorphism induced by the map q.

The *stabilizer* Γ_a of a point $a \in M$ under the action τ is the subgroup consisting of all elements $\mu \in \Gamma$ such that $\tau\mu(a) = a$. The action τ is called *transitive* if, for any pair of points a, $b \in M$, there exists an element $\mu \in \Gamma$ such that $\tau\mu(a) = b$.

The following proposition is obvious:

Proposition 2. *The following properties hold.*

1. *An action τ of a group Γ is transitive if and only if the stabilizers of any two points a, $b \in M$ are conjugate.*

   The image of the group Γ under a transitive action τ is isomorphic to the quotient group

$$\Gamma / \bigcap_{\mu \in \Gamma} \mu \Gamma_a \mu^{-1}.$$

2. *There exists a transitive action of the group* Γ *with a given stabilizer of some point, and this transitive action is unique up to equivalence.*

Thus transitive actions of a group Γ are described by pairs of groups.

The pair of groups
$$[\Gamma, \Gamma_a],$$
where Γ_a is the stabilizer of some point a under a transitive action τ of the group Γ, is called the *monodromy pair of the point* a with respect to the action τ.

The group
$$\tau(\Gamma) \sim \Gamma / \bigcap_{\mu \in \Gamma} \mu \Gamma_a \mu^{-1}$$
is called *the monodromy group of the pair* $[\Gamma, \Gamma_a]$.

The A-monodromy homomorphism τ gives rise to a transitive action of the fundamental group $\pi_1(S^2 \setminus A)$ on the set F_a of branches of the function f over the point a.

The monodromy pair of the germ f_a with respect to the action τ is called the *monodromy pair of the germ* f_a *with the forbidden set* A.

The monodromy pair of the germ f_a with respect to the action of the closed monodromy group is called the *closed monodromy pair of the germ* f_a.

Different germs of the $\mathcal{S}$-function f have isomorphic monodromy pairs with the forbidden set A, hence it makes sense to speak of the monodromy pair with the forbidden set A and the closed monodromy pair of the $\mathcal{S}$-function f.

The closed monodromy pair of the $\mathcal{S}$-function f will be denoted by $[f]$.

ALMOST NORMAL FUNCTIONS

Definition 4. *A pair of groups*

$$[\Gamma, \Gamma_0], quad \Gamma_0 \subseteq \Gamma$$

is called an almost normal pair *if there exists a finite set* $P \subset \Gamma$ *such that*

$$\bigcap_{\mu \in \Gamma} \mu \Gamma_0 \mu^{-1} = \bigcap_{\mu \in P} \mu \Gamma_0 \mu^{-1}.$$

Proposition 3 (on a discrete action)**.** *The image $\tau(\Gamma)$ of the group Γ under a transitive action*

$$\tau : \Gamma \to S(M)$$

is a discrete subgroup of $S(M)$ if and only if the monodromy pair $[\Gamma, \Gamma_0]$ of some element $x_0 \in M$ is almost normal.

Proof. Let the group $\tau(\Gamma)$ be discrete. Let $\overline{P}$ denote a finite subset of the set M such that the neighborhood $U_{\overline{P}}$ of the identity contains no transformations of the group $\tau(\Gamma)$ different from the identity.

This means that the intersection $\bigcap_{x\in\overline{P}} \Gamma_x$ of the stabilizers of points $x \in \overline{P}$ acts trivially on the set M, i.e.

$$\bigcap_{x\in\overline{P}} \Gamma_x \subseteq \bigcap_{\mu\in\Gamma} \mu\Gamma_0\mu^{-1}.$$

The groups Γ_x are conjugate to the group Γ_0, hence we can choose a finite set $P \subset \Gamma$ such that

$$\bigcap_{\mu \in P} \mu \Gamma_0 \mu^{-1} = \bigcap_{\mu \in \Gamma} \mu \Gamma_0 \mu^{-1}.$$

The converse statement can be proved similarly. $\square$ $\square$

Definition 5. *A $\mathcal{S}$-function f is called* almost normal *if its monodromy group is discrete. From the lemma, it follows that the function f is almost normal if and only if its closed monodromy pair $[f]$ is almost normal.*

A *differential rational function* of several functions is defined to be a rational function of these functions and their derivatives.

Lemma 2 (on finitely generated functions)**.** *Suppose that every germ of a $\mathcal{S}$-function f over the point a is a differential rational function of finitely many fixed germs of f over a.*

Then the function f is almost normal.

Proof. Indeed, if, under continuation along a closed path, the specified germs of the function are unchanged, then a differential rational function of them is also unchanged. □

From the lemma on finitely generated functions, it follows that any solution of a linear differential equation with rational coefficients is an almost normal function. The same is also true for many other functions appearing naturally in differential algebra.

CLASSES OF GROUP PAIRS

We will describe how the closed monodromy pairs of functions are transformed under compositions, integrations, differentiations, etc. To this end, we will need to introduce some notions concerning pairs of groups (group pairs).

A *group pair* always means a pair consisting of a group and its subgroup.

We will identify a group with the group pair consisting of this group and its trivial subgroup.

DEFINITION. A collection $\mathcal{L}$ of group pairs will be referred to as an *almost complete class of group pairs* if

1. for every group pair

$$[\Gamma, \Gamma_0] \in \mathcal{L}, \;\; \Gamma_0 \subseteq \Gamma,$$

   and any homomorphism $\tau : \Gamma \to G,$ where G is any group, the group pair $[\tau\Gamma, \tau\Gamma_0]$ is also contained in $\mathcal{L}$;

2. for every group pair

$$[\Gamma, \Gamma_0] \in \mathcal{L}, \ \Gamma_0 \subseteq \Gamma,$$

and any homomorphism $\tau : G \to \Gamma$, where G is any group, the group pair $[\tau^{-1}\Gamma, \tau^{-1}\Gamma_0]$ is also contained in $\mathcal{L}$,

3. for every group pair

$$[\Gamma, \Gamma_0] \in \mathcal{L}, \ \Gamma_0 \subseteq \Gamma,$$

and a group G equipped with a T_2-topology and containing the group Γ, $\Gamma \subseteq G$, the group pair $[\overline{\Gamma}, \overline{\Gamma}_0]$ is also contained in $\mathcal{L}$, where $\overline{\Gamma}$, $\overline{\Gamma}_0$ are the closures of the groups Γ, Γ_0 in the group G.

Definition. An almost complete class $\mathcal{M}$ of group pairs will be referred to as a *complete class of group pairs*, if

1. for every group pair

$$[\Gamma, \Gamma_0] \in \mathcal{M}$$

and a group

$$\Gamma_1, \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma,$$

the group pair $[\Gamma, \Gamma_1]$ is also contained in $\mathcal{M}$,

2. for every two group pairs

$$[\Gamma, \Gamma_1], \quad [\Gamma_1,\ \Gamma_2] \in \mathcal{M},$$

the group pair $[\Gamma, \Gamma_2]$ is also contained in $\mathcal{M}$.

The minimal almost complete and, respectively, complete classes of group pairs containing a fixed set $\mathcal{B}$ of group pairs are denoted by $\mathcal{L}\langle\mathcal{B}\rangle$ and, respectively, by $\mathcal{M}\langle\mathcal{B}\rangle$.

Lemma 3. *The following properties hold.*

1. *If the monodromy group of the pair $[\Gamma, \Gamma_0]$ is contained in some complete class $\mathcal{M}$ of pairs, then the pair $[\Gamma, \Gamma_0]$ is also contained in $\mathcal{M}$.*

2. *If an almost normal pair $[\Gamma, \Gamma_0]$ is contained in some complete class $\mathcal{M}$ of group pairs, then its monodromy group is also contained in $\mathcal{M}$.*

Let us give a proof of the second statement.

Let $\Gamma_i, \quad i = 1, \ldots, n$ be a finite number of subgroups conjugate to Γ_0 such that

$$\bigcap_{i=1}^{n} \Gamma_i = \bigcap_{\mu \in \Gamma} \mu \Gamma_0 \mu^{-1}.$$

The pair $[\Gamma, \Gamma_i]$ is isomorphic to the pair $[\Gamma, \Gamma_0]$, hence $[\Gamma, \Gamma_i] \in \mathcal{M}$. Let

$$\tau : \Gamma_2 \to \Gamma$$

be the inclusion homomorphism, then

$$\tau^{-1}(\Gamma_1) = \Gamma_2 \bigcap \Gamma_1,$$

hence $[\Gamma_2, \Gamma_2 \bigcap \Gamma_1] \in \mathcal{M}$. The class $\mathcal{M}$ contains the pairs

$$[\Gamma, \Gamma_2] \quad \text{and} \quad [\Gamma_2, \Gamma_2 \bigcap \Gamma_1],$$

therefore,

$$[\Gamma, \Gamma_1 \bigcap \Gamma_2] \in \mathcal{M}.$$

Continuing this argument, we obtain that the class $\mathcal{M}$ contains the pair $[\Gamma, \bigcap_{i=1}^{n} \Gamma_i]$ and, together with it, the group

$$\Gamma / \bigcap_{\mu \in \Gamma} \mu \Gamma_0 \mu^{-1}.$$

Proposition 4. *An almost complete class of pairs $\mathcal{L}$ contains the closed monodromy pair*

$$[f]$$

of a $\mathcal{S}$-function f if and only if this class contains the monodromy pair of the function f with a forbidden set A.

Proof. Let $[\Gamma, \Gamma_0]$ be the monodromy pair of the function f with a forbidden set A.

Then $[f] = [\overline{\Gamma}, \overline{\Gamma}_0]$. Hence every almost complete class $\mathcal{L}$ containing the pair $[\Gamma, \Gamma_0]$ also contains the pair $[f]$. Conversely, if $[\overline{\Gamma}, \overline{\Gamma}_0]$ is contained in the class $\mathcal{L}$, then $[\Gamma, \Gamma_0] \in \mathcal{L}$.

Indeed, the topology on the permutation group is such that $\Gamma_0 = \Gamma \bigcap \overline{\Gamma}_0$. Hence, the pair $[\Gamma, \Gamma_0]$ is the preimage of the pair $[\overline{\Gamma}, \overline{\Gamma}_0]$ under the inclusion of the group Γ into its closure. $\square$ $\square$

THE MAIN THEOREM

Let us formulate and prove the main theorem of the topological Galois theory.

Theorem 4. *A class of $\mathcal{S}$-functions $\hat{\mathcal{M}}$ consisting of $\mathcal{S}$-functions whose closed monodromy pairs lie in some complete class $\mathcal{M}$ of* *pairs, is stable under the differentiation, composition and meromorphic operations.*

Furthermore, if the class $\mathcal{M}$ *contains*

1. *the additive group* $\mathbb{C}$ *of complex numbers, then the class* $\hat{\mathcal{M}}$ *is stable under the integration,*
2. *the permutation group* $S(k)$ *of* k *elements, then the class* $\hat{\mathcal{M}}$ *is stable under solving algebraic equations of degrees at most* k*.*