Lecture 20, November 24

FUNCTIONAL DIFFERENTIAL FIELDS

Let us now turn to functional differential fields. We will be dealing with this particular type of fields (although some results can be easily extended to abstract differential fields).

Example 1. *Let K be a subfield in the field of all meromorphic functions on a connected domain U of the Riemann sphere. Suppose that K contains all complex constants and is stable under differentiation (i.e. if $f \in K$, then $f' \in K$). Then K provides an example of a functional differential field.*

Let us now give a general definition.

Definition 1. *Let V, α be a pair consisting of a connected Riemann surface V and a meromorphic 1-form α on it. Using α one can define a derivation on the field of meromorphic functions on V by the following formula:*

$$f' = df/\alpha$$

(the ratio of two meromorphic 1-forms is a well-defined meromorphic function).

A functional differential field is any subfield of the field of meromorphic functions on v, containing all complex constants and closed under above differentiation.

The following construction helps to *extend* functional differential fields.

Let K be a subfield in the field of meromorphic functions on a connected Riemann surface V equipped with a meromorphic form α, and suppose that the subfield is invariant under the derivation

$$f' = \frac{df}{\alpha},$$

(i.e., if $f \in K$, then $f' \in K$).

Consider a connected Riemann sufrace w and together with a non-constant analytic map

$$\pi : W \to V.$$

Fix the the following form on W:

$$\beta = \pi^* \alpha.$$

The differential field F of all meromorphic functions on W with the differentiation

$$\varphi' = \frac{d\varphi}{\beta}$$

contains the differential subfield

$$\pi^* K$$

consisting of functions of the form

$$\pi^* f, \quad where \quad f \in K.$$

The differential field $\pi^* K$ is isomorphic to the differential field K, and it lies in the differential field F.

For a suitable choice of the surface W, an extension of the field $\pi^* K$, which is isomorphic to K, can be done within the field F.

Suppose that we need to extend the field K, say, by an integral y of some function $f \in K$.

This can be done in the following way. Consider the covering of the Riemann surface V by the Riemann surface W of an indefinite integral

$$y = \int f\alpha$$

of the form $f\alpha$.

By the very definition of the Riemann surface W, there exists a natural projection

$$\pi : W \to V,$$

and the function y is a single-valued meromorphic function on the surface W. The differential field F of meromorphic functions on W with the differentiation

$$\varphi' = d\varphi/\pi^*\alpha$$

contains the element y as well as the field $\pi^* K$ isomorphic to K.

That is why the extension $\pi^* K\langle y\rangle$ is well defined as a subfield of the differential field F. We mean this particular construction of the extension whenever we talk about extensions of functional differential fields.

The same construction allows to adjoin

a logarithm, an exponential,

an integral or an exponential of integral of any function f from a functional differential field K to K.

Similarly, for any functions $f_1, \ldots, f_n \in K$, one can adjoin a solution y of an algebraic equation

$$y^n + f_1 y^{n-1} + \cdots + f_n = 0$$

or all the solutions $y_1, \ldots, y_n$ of this equation to K

(the adjunction of all the solutions $y_1, \ldots, y_n$ can be implemented on the Riemann surface of the vector-function

$$\mathbf{y} = y_1, \ldots, y_n).$$

In the same way, for any functions $f_1, \ldots, f_{n+1} \in K$, one can adjoin the n-dimensional $\mathbb{C}$ affine space of all solutions of the linear differential equation

$$y^{(n)} + f_1 y^{(n-1)} + \cdots + f_n y + f_{n+1} = 0$$

to K.

(Recall that a germ of any solution of this linear differential equation admits an analytic continuation along a path on the surface V not passing through the poles of the functions $f_1, \ldots, f_{n+1}$.)

Thus, *all above–mentioned extensions of functional differential fields can be implemented without leaving the class of functional differential fields.*

When talking about extensions of functional differential fields, we always mean this particular procedure.

The differential field of all complex constants and the differential field of all rational functions of one variable can be regarded as differential fields of functions defined on the Riemann sphere.

Let us restate the Theorem about equivalents of two definitions of the Liouvillian classes of functions using definitions from abstract differential algebra and the construction of extensions of functional differential fields.

Theorem 1. *A function of one complex variable (possibly multi-valued) belongs to:*

1. *the class of elementary functions if and only if it belongs to some elementary extension of the field of all rational functions of one variable;*
2. *the class of generalized elementary functions if and only if it belongs to some generalized elementary extension of the field of rational functions;*
3. *the class of functions representable by quadratures if and only if it belongs to some Liouville extension of the field of all complex*

constants;

4. *the class of functions representable by k-quadratures if and only if it belongs to some k-Liouville extension of the field of all complex constants;*
5. *the class of functions representable by generalized quadratures if and only if it belongs to a generalized Liouville extension of the field of all complex constants.*

MONODROMY GROUPS OF BASIC FUNCTIONS RELATED TO REPRESENTABILITY BY RADICAL

It is easy to compute the monodromy groups of basic functions elementary functions.

The monodromy groups of constants, of the independent variable and of the exponential functions are trivial (i.e. contain the identity element only), since these functions are single valued.

Proposition 1. *The monodromy group of the logarithm*

$$y = \ln x$$

is isomorphic to the additive group of integral numbers $\mathbb{Z}$*.*

Proof. Indeed, the multivalued function $y = \ln x$ for $x \neq 0$ is defined by the following formulas:

$$Re \ln x = \ln |x|, \quad \Im lnx = \alpha,$$

where α is the argument of x which is defined up to addition $2\pi k$. Thus $\ln x$ has no singular points in the domain

$$\mathbb{C}^* = \overline{\mathbb{C}} \setminus \{0, \infty\},$$

whose fundamental group is isomorphic to $\mathbb{Z}$.

A generator of the group $\pi_1(\mathbb{C}^*, 1)$ is represented by the loop

$$\gamma(t) = \exp(2\pi i t), \quad 0 \leq t \leq 1.$$

The function $y = \ln x$ has germs $\ln x + 2\pi k$ where $\ln x$ is the Taylor series of the real function $y = \ln x$ at the point 1 and $k \in \mathbb{Z}$.

The analytic continuation of the germ $y_k = \ln x + 2pik$ along the curve γ takes the germ y_k to the germ y_{k+1}.

Thus the monodromy group of $\ln x$ is isomorphic to the additive group of integral numbers.

$\square$

Elementary functions can be constructed via one valued functions and the function $\ln x$ having commutative monodromy group.

We know that the functions representable by one valued rational functions on the functions $y = x^{1/k}$ having commutative monogromny groups have solvable monodromy groups.

By analogy one can state the following natural conjecture.

Conjecture 1. *For any elementary function the monodromy groups is well defined and it is solvable.*

It turn out that the conjecture is true. Nevertheless a way how the Riemann surface of an elementary function covers the complex line could be much more complicated than one can expected.

ONE DIMENSIONAL TOPOLOGICAL GALOIS THEORY

Not only algebraic functions have monodromy groups. It is also defined for basic elementary functions and many more functions, for which the Galois group does not make sense

For such functions, it is natural to use the monodromy group instead of the Galois group for proving that a function does not belong to a certain Liouville class.

This approach is implemented in the topological version of Galois theory.

Let us give an example that shows some difficulties that we need to overcome on this way.

PROGRAM:

I. Find a wide class of functions which is closed under classical operations, such that for all functions from the class the monodromy group is well defined.

II. Use the monodromy group within this class instead of the Galois group.

Consider the elementary function f defined by the following formula:

$$f(z) = \ln\left(\sum_{j=1}^{n} \lambda_j \ln(z - a_j)\right),$$

where

$$a_j, \quad j = 1, \ldots, n$$

are different points in the complex line, and

$$\lambda_j, \quad j = 1, \ldots, n$$

are complex constants.

Let Λ denote the additive subgroup of complex numbers generated by the constants

$$\lambda_1, \quad \ldots, \lambda_n.$$

It is clear that if $n > 2$, then for almost every collection of constants $\lambda_1, \ldots, \lambda_n$, the group Λ is everywhere dense in the complex line.

Proposition 2. *If the group Λ is dense in the complex line, then the elementary function f has a dense set of logarithmic ramification points.*

Proof. Let g_a be one of the germs of the function g defined by the formula
$$g(z) = \sum_{j=1}^{n} \lambda_j \ln(z - a_j)$$
at a point
$$a, a \neq a_j, \quad j = 1, \ldots, n.$$
A loop around the points
$$a_1, \ldots, a_n$$
adds the number
$$2\pi i\lambda$$
to the germ g_a, where λ is an element of the group Λ.

Conversely, every germ

$$g_a + 2\pi i\lambda, \quad \textit{where} \quad \lambda \in \Lambda,$$

can be obtained from the germ g_a by the analytic continuation along some loop.

Let U be a small neighborhood of the point a, and

$$G : U \to \mathbb{C}$$

an analytic function, whose germ at the point a is g_a.

The image V of the domain U under the map $G : U \to \mathbb{C}$ is open. Therefore, in the domain V, there is a point of the form $2\pi i\lambda$, where $\lambda \in \Lambda$.

The function

$$G - 2\pi i\lambda$$

is one of the branches of the function g over the domain U, and the zero set of this branch in the domain U is nonempty.

Hence, one of the branches of the function

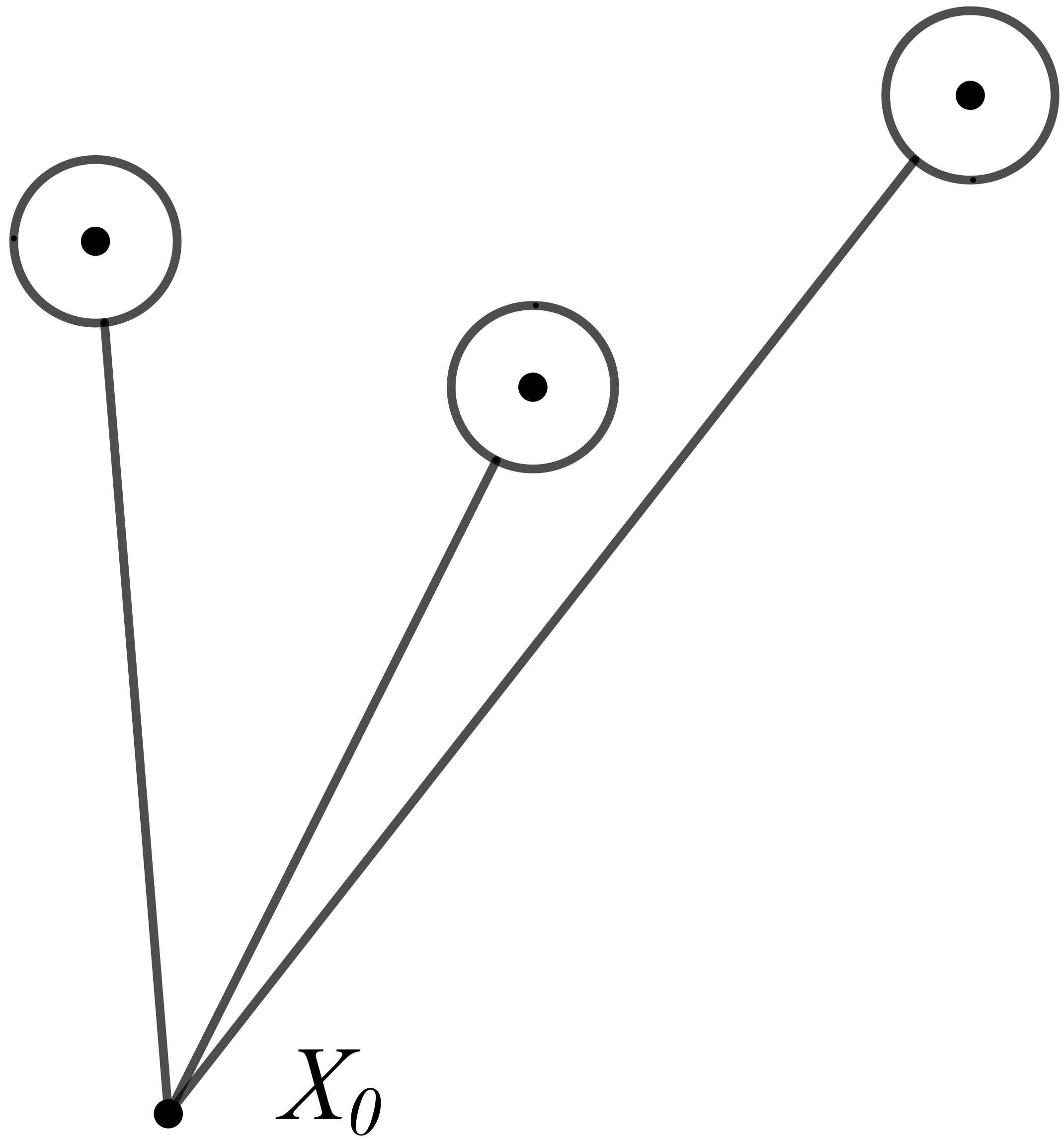
$$f = \ln g$$

has a logarithmic ramification point in U. $\square$ $\square$

It is not hard to verify that, under the assumptions of the proposition, the monodromy group of the function f has the cardinality of the continuum (this is not surprising: the fundamental group

$$\pi_1(S \setminus A)$$

has obviously the cardinality of the continuum provided that A is a countable dense set in the Riemann sphere).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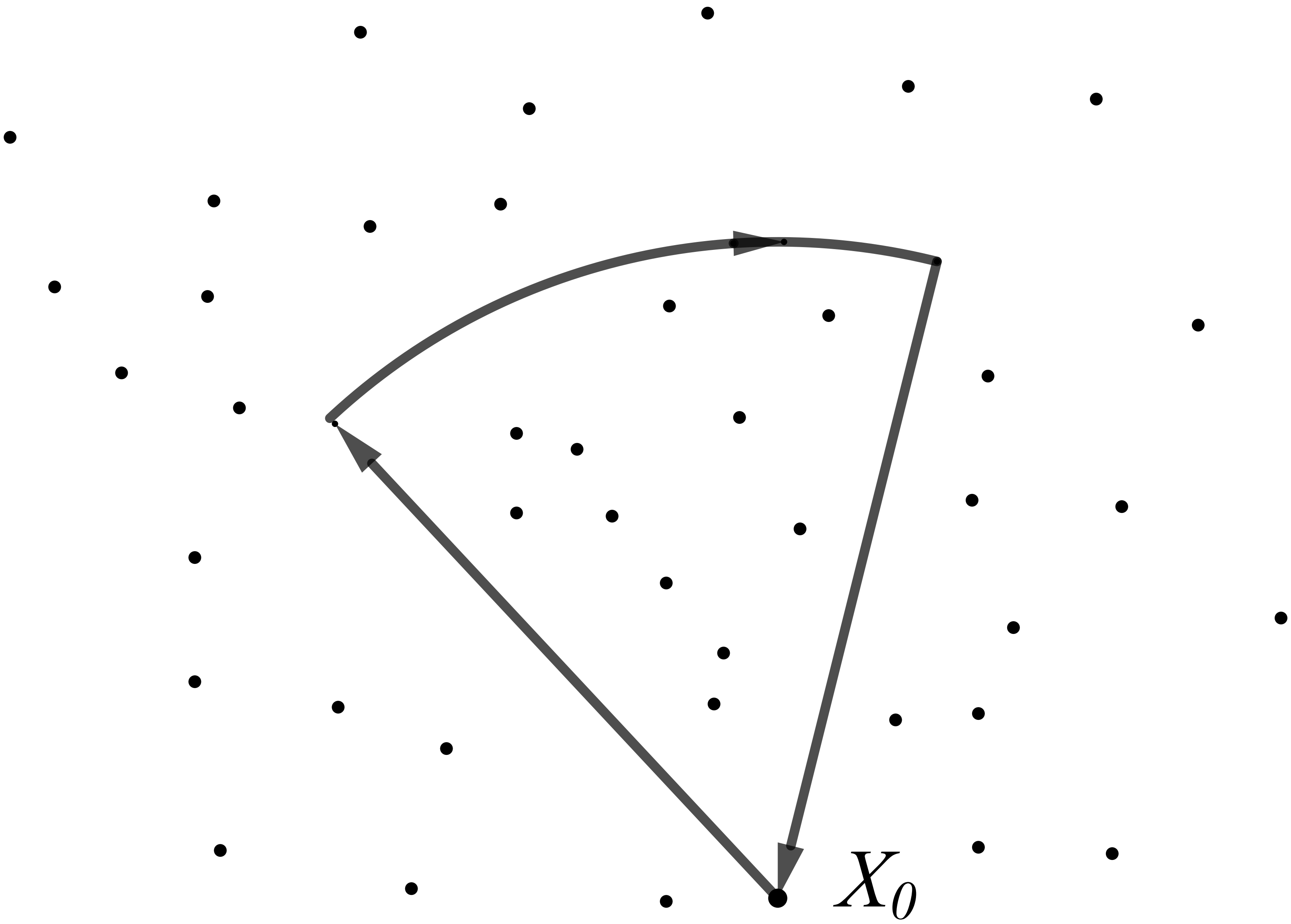
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One can also prove that the image of the fundamental group $\pi_1(S^2 \setminus \{A \cup b\})$ of the complement of the set

$$A \cup b,$$

where $b \notin A$, in the permutation group of branches of the function f is a proper subgroup of the monodromy group of f.

The fact that the removal of one extra point can change the monodromy group, makes all proofs more complicated.

Thus even simplest elementary functions can have dense singular sets and monodromy groups of cardinality of the continuum.

Nevertheless, the set of singular points of an elementary function is at most countable, and its monodromy group is solvable.

If a function does not satisfy these restrictions, then it cannot be elementary. There exist similar topological obstructions to the membership of a function of one complex variable in other Liouvillian classes of functions.

We now proceed with a detailed description of this geometric approach to the problem of solvability.

FUNCTIONS WITH AT MOST COUNTABLE SINGULAR SETS

We define a broad class of functions of one complex variable needed in the construction of the topological version of Galois theory.

FORBITEN SETS

We now define the class of functions that will be dealt with. A multivalued analytic function of one complex variable is called a $\mathcal{S}$-*function*, if the set of its singular points is at most countable.

Let us make this definition more precise.

Definition 2. *Two regular germs f_a and g_b defined at points a and b of the Riemann sphere*

$$S^2 = \mathbb{C} + \{\infty\}$$

are called equivalent *if the germ g_b is obtained from the germ f_a by the regular (analytic) continuation along some path.*

Each germ g_b equivalent to the germ f_a is also called a regular germ of the multivalued analytic function f generated by the germ f_a.

Definition 3. *A point $b \in S^2$ is said to be* singular *for the germ f_a if there exists a path*

$$\gamma : [0, 1] \to S^2, \quad \gamma(0) = a, \gamma(1) = b$$

such that the germ has no regular continuation along this path, *but for any t, $0 \leq t < 1$, it admits a regular continuation along the truncated path*

$$\gamma : [0, t] \to S^2.$$

It is easy to see that equivalent germs have the same set of singular points.

Definition 4. *A regular germ is called a $\mathcal{S}$-*germ*, if the set of its singular points is at most countable.*

A multivalued analytic function is called a $\mathcal{S}$-function if each its regular germ is a $\mathcal{S}$-germ.

We will need a lemma that allows to release a plane path from a countable set by a small deformation.

Lemma 2 (on releasing a path from a countable set)**.** *Let A be at most countable subset of the plane of complex numbers,*

$$\gamma : [0,1] \to \mathbb{C}$$

be a continuous path, and φ be a continuous positive function on the interval $0 < t < 1$.

Then there exists a path $\hat{\gamma} : [0,1] \to \mathbb{C}$ such that for $0 < t < 1$ we have

$$\hat{\gamma}(t) \notin A$$

and

$$|\gamma(t) - \hat{\gamma}(t)| < \varphi(t).$$

A high-technology proof of the lemma is as follows.

In the functional space of paths $\overline{\gamma}$ close to the path γ, say, satisfying the inequality

$$|\gamma(t) - \overline{\gamma}(t)| \leq \varphi(t)/2,$$

the paths avoiding one particular point of A form an open dense set.

The intersection of countably many open dense sets in such functional spaces is nonempty (it is easy to see that the space is complete).

Let us give an elementary proof of the lemma (almost verbatim, it carries over to a more general case when the set A is uncountable but has zero Hausdorff length).

Proof. Let us first construct a continuous broken line $\overline{\gamma}$ with infinitely many edges such that its vertices do not belong to A and

$$|\gamma(t) - \overline{\gamma}(t)| < \frac{1}{2}\varphi(t).$$

Such broken line can be constructed since the complement of the set A is dense.

Let us show how to change each edge $[p, q]$ of the broken line $\overline{\gamma}$ to make it avoid the set A.

Take an interval $[p, q]$. Let m be the perpendicular bisector of it. Consider broken lines with 2 edges

$$[p, b], [b, q], \quad \text{where} \quad b \in m,$$

and the point b is sufficiently close to the interval. These broken lines intersect by the endpoints p, q only, and their cardinality is that of the continuum. Therefore, there exists a broken line among them that does not intersect the set A. Changing each edge of the initial broken line in this way, we obtain the desired curve. $\square$

Besides the set of singular points, it is also convenient to consider other sets such that the function admits analytic continuation everywhere in the complement.

Definition 5. *An at most countable set A is called a* forbidden set *for a regular germ f_a if the germ f_a admits a regular continuation along any path*

$$\gamma(t), \quad \gamma(0) = a,$$

never intersecting the set A except possibly at the initial moment.

Theorem 3 (on a forbidden set). *An at most countable set is a forbidden set of a germ if and only if it contains the set of its singular points.*

In particular, a germ has a forbidden set if and only if it is a germ of a $\mathcal{S}$-function.

Proof. Suppose that there exists a singular point b of a germ f_a that does not lie in a forbidden set A of this germ.

By definition, there must be a path

$$\gamma : [0,1] \to S^2, \quad \gamma(0) = a, \quad \gamma(1) = b,$$

such that there is no regular continuation of the germ f_a along it, but the germ can be continued up to any $t < 1$.

Without loss of generality, we can assume that the points a, b and the path $\gamma(t)$ lie in the finite part of the Riemann sphere, i. e.

$$\gamma(t) \neq \infty$$

for $0 \le t \le 1$.

Let $R(t)$ denote the radius of convergence of the series $f_{\gamma(t)}$ obtained by continuation of the germ f_a along the path $\gamma : [0, t] \to S^2$.

The function $R(t)$ is continuous on the half-open interval $[0, 1)$.

By the lemma, there exists a path

$$\hat{\gamma}(t), \quad \hat{\gamma}(0) = a, \quad \hat{\gamma}(1) = b,$$

such that

$$|\gamma(t) - \hat{\gamma}(t)| < \frac{1}{3} R(t)$$

and $\hat{\gamma}(t) \notin A$ for $t > 0$.

By the assumption, the germ f_a admits a continuation along the path

$$\hat{\gamma}$$

up to the point 1.

But it follows easily that the germ f_a admits a continuation along the path

$$\gamma.$$

The contradiction proves that the singular set of the germ f_a is contained in every forbidden set of this germ.

The converse statement (a countable set containing the singular set of the germ is forbidden for the germ) is obvious. $\square$ $\square$