Lecture 18, November 17

DEFINING CLASSES OF FUNCTIONS BY THE LISTS OF BASIC FUNCTIONS AND OF ADMISSIBLE OPERATIONS

A class of functions can be introduced by specifying a list of *basic functions* and a list of *admissible operations*.

Given the two lists, the class of functions is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations.

Later, we define Liouvillian classes of functions in exactly this way.

Liouvillian classes of functions, which appear in the problems of solvability in finite terms, contain multivalued functions. Thus the basic terminology should be made clear.

We work with multivalued functions "globally",
which leads to a more general understanding of classes of functions defined by lists of basic functions and of admissible operations.

In this global version, a multivalued function is regarded as a single entity.

Operations on multivalued functions can be defined. The result of such an operation is a set of multivalued functions; every element of this set is called a function obtained from the given functions by the given operation. A *class of functions* is defined as the set of all (multivalued) functions that can be obtained from the basic functions by repeated application of admissible operations.

Let us define, for example, the sum of two multivalued functions of one variable.

Definition 1. *Take an arbitrary point a in the complex line, any germ f_a of an analytic function f at the point a and any germ g_a of an analytic function g at the same point a.*

We say that the multivalued function φ generated by the germ

$$\varphi_a = f_a + g_a$$

is representable as the sum of the functions f *and* g.

Example 1. *It is easy to see that exactly two functions are representable in the form*

$$\sqrt{x} + \sqrt{x},$$

namely,

$$f_1 = 2\sqrt{x} \quad \textit{and} \quad f_2 \equiv 0.$$

Other operations on multivalued functions are defined in exactly the same way.

For a class of multivalued functions, being stable under addition means that, together with any pair of its functions, this class contains all functions representable as their sum.

The same applies to all other operations on multivalued functions understood in the same sense as above.

In the definition given above, not only the operation of addition plays a key role but also the operation of analytic continuation hidden in the notion of multivalued function.

Indeed, consider the following example.

Example 2. *Let f_1 be an analytic function defined on an open subset U of the complex line $\mathbb{C}^1$ and admitting no analytic continuation outside of U, and let f_2 be an analytic function on U given by the formula $f_2 = -f_1$.*

According to our definition, the zero function is representable in the form

$$f_1 + f_2$$

on the entire complex line.

By the commonly accepted viewpoint, the equality

$$f_1 + f_2 = 0$$

holds inside the region U but not outside.

Working with multivalued functions globally, we do not insist on the existence of *a common region*, where all necessary operations would be performed on single-valued branches of multivalued functions.

A first operation can be performed in a first region, then a second operation can be performed in a second, different region on analytic continuations of functions obtained on the first step.

In essence, this more general understanding of operations is equivalent to including analytic continuation to the list of admissible operations on the analytic germs.

For single variable functions, it is possible to obtain topological obstructions even with this more general understanding of operations on multivalued analytic functions.

In the sequel, when considering topological obstructions to the membership of an analytic function in a certain class, we will always mean this global definition of the function class via lists of basic functions and admissible operations.

For functions of many variables, things do not work in this general setting, and one is forced to adopt a more restrictive formulation dealing with the germs of functions. It is no less (and perhaps even more) natural, however.

In this course we will discuss the theory for functions of single variable only.

LIOUVILLIAN CLASSES OF FUNCTIONS

We define Liouvillian classes of single variable functions by lists of basic functions and admissible operations.

FUNCTIONS OF ONE VARIABLE REPRESENTABLE BY RADICALS

List of basic functions:
all complex constants, an independent variable x.
List of admissible operations:
arithmetic operations and the operation of taking the n-th root $\sqrt[n]{f}$, $n = 2, 3, \ldots,$ of a given function f.

The function

$$g(x) = \sqrt[3]{5x + 2\sqrt[2]{x}} + \sqrt[7]{x^3 + 3}$$

gives an example of a function representable by radicals.

The famous problem of solvability of equations by radicals is related to this class. Consider an algebraic equation

$$y^n + r_1 y^{n-1} + \cdots + r_n = 0,$$

in which r_i are rational functions of one variable.

As we studied before, a complete answer to the question of solvability of such equations by radicals is given by Galois theory.

To define other classes, we will need the list of *basic elementary functions.* In essence, this list contains functions that are studied in high-school

(and/or in pre-calculus college courses) and which are frequently used in pocket calculators.

LIST OF BASIC ELEMENTARY FUNCTIONS

1. All complex constants and an independent variable x.

2. The exponential, the logarithm and the power x^α, where α is any complex constant.

3. Trigonometric functions: sine, cosine, tangent, cotangent.

4. Inverse trigonometric functions: arcsine, arccosine, arctangent, arccotangent.

LIST OF CLASSICAL OPERATIONS

1) composition: $f, g \in L \Rightarrow f \circ g \in L$;

2) arithmetic operations: $f, g \in L \Rightarrow f \pm g, f \times g, f/g \in L$;

3) differentiation: $f \in L \Rightarrow f' \in L$;

4) integration: $f \in L$ and $y' = f$, i.e. $y = C + \int^{x} f(t)dt \Rightarrow y \in L$;

5) extension by exponent of integral: $f \in L$ and $y' = fy$, i.e. $y = C \exp \int^{x} f(t)dt \Rightarrow y \in L$;

6) algebraic extension: $f_1, \dots, f_n \in L$ and $y^n + f_1 y^{n-1} + \dots + f_n = 0 \Rightarrow y \in L$;

7) exponent: $f \in L$ and $y' = f'y$, i.e. $y = C \exp f \Rightarrow y \in L$;

8) logarithm: $f \in L$ and $dy = df/f$, i.e. $y = C + \ln f \Rightarrow y \in L$;

9) meromorphic operation: if $F : \mathbb{C}^n \to \mathbb{C}$ is a meromorphic function, $f_1 \dots, f_n \in L$, and $y = F(f_1, \dots, f_n) \Rightarrow y \in L$.

The operations 2) and 7) are meromorphic operations.

ELEMENTARY FUNCTIONS

LIST OF BASIC FUNCTIONS:

basic elementary functions.

LIST OF ADMISSIBLE OPERATIONS:

compositions, arithmetic operations, differentiation.

All elementary functions are given by formulas, e.g. the following:

$$f(x) = \operatorname{arctg}(\exp(\sin x) + \cos x).$$

FUNCTIONS REPRESENTABLE BY QUADRATURES

LIST OF BASIC FUNCTIONS:

basic elementary functions.

LIST OF ADMISSIBLE OPERATIONS:

compositions, arithmetic operations, differentiation, integration.

For example, the elliptic integral

$$f(x) = \int_{x_0}^{x} \frac{dt}{\sqrt{P(t)}},$$

where P is a polynomial, is representable by quadratures. But $f(x)$ is not elementary function if $\deg P \geq 3$ and P has no multiple roots.

GENERALIZED ELEMENTARY FUNCTIONS

This class of functions is defined in the same way as the class of elementary functions. We only need to add the operation of solving algebraic equations to the list of admissible operations.

FUNCTIONS REPRESENTABLE BY GENERALIZED QUADRATURES

This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations to the list of admissible operations.

Let us now define two more classes of functions similar to Liouvillian classes.

FUNCTIONS REPRESENTABLE BY k-RADICALS

This class of functions is defined in the same way as the class of functions representable by radicals.

We only need to add the operation of solving algebraic equations of degree $\leq k$ to the list of admissible operations.

FUNCTIONS REPRESENTABLE BY k-QUADRATURES

This class of functions is defined in the same way as the class of functions representable by quadratures.

We only need to add the operation of solving algebraic equations of degree at most k to the list of admissible operations.

Lemma 1. *Basic elementary functions can be expressed through the exponentials and the logarithms with the help of complex constants, arithmetic operations and compositions.*

Proof. For a power function x^α, the required expression is given by the equality
$$x^\alpha = \exp(\alpha \ln x).$$

For the trigonometric functions, the required expressions follow from Euler's formula
$$e^{a+bi} = e^a(\cos b + i \sin b).$$

For real values of x, we have
$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

and

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}).$$

By analyticity, the same formulas remain true for all complex values of x.

The tangent and the cotangent functions are expressed through the sine and the cosine.

Let us now show that, for all real x, the equality

$$\operatorname{arctg} x = \frac{1}{2i} \ln z$$

holds, where

$$z = \frac{1 + ix}{1 - ix}.$$

Obviously,

$$|z| = 1,\ \arg z = 2\arg(1+ix),\ \mathrm{tg}(\arg(1+ix)) = x,$$

which proves the desired equality. By analyticity, the same equality also holds for all complex values of x.

The remaining inverse trigonometric functions can be expressed through the arctangent. Namely,

$$\mathrm{arcctg} x = \frac{\pi}{2} - \mathrm{arctg} x,\ \arcsin x = \mathrm{arctg}\frac{x}{\sqrt{1-x^2}},$$

$$\arccos = \frac{\pi}{2} - \arcsin x.$$

The square root that appears in the expression for the function arcsin can be expressed through the exponential and the logarithm: $x^{1/2} = \exp\left(\frac{1}{2}\ln x\right)$. The lemma is proved. $\square$ $\square$