Lecture 14, October 27

INTRODUCTION TO TOPOLOGICAL GALOIS THEORY

ON REPRESENTABILITY OF ALGEBRAIC FUNCTIONS BY RADICALS

I am going to sketch a simplest version of the topological Galois theory. I will return back to a detailed careful presentation of material later. Now I will skip many details. My goal is a vosial presentation of key ideas of the theory

More precisely I will present a self contained simple proof of the classical criteria for representability of algebraic functions of several complex variables by radicals.

The presentation also contains a criteria for representability of algebroidal functions by composition of single-valued analytic functions and radicals, and a result related to the 13-th Hilbert problem.

Consider an algebraic equation

$$P_n y^n + P_{n-1} y^{n-1} + \cdots + P_0 = 0, \tag{1}$$

whose coefficients $P_n, \ldots, P_0$ are polynomials of N complex variables $x_1, \ldots, x_N$.

Camille Jordan discovered that the Galois group of the equation (1) over the field $\mathcal{R}$ of rational functions of $x_1, \ldots, x_N$ has a topological meaning (see theorem 3 below):

It is isomorphic to the *monodromy group* of the equation (1).

According to the Galois theory, equation (1) is solvable by radicals over the field R if and only if its Galois group is solvable. If the equation (1) is irreducible it defines a multivalued algebraic function $y(x)$.

The Galois theory and Theorem 3 imply the following criteria for representability of an algebraic function by radicals, which consists of two statements:

1) *If the monodromy group of an algebraic function* $y(x)$ *is solvable, then* $y(x)$ *is representable by radicals.*

2) *If the monodromy group of an algebraic function* $y(x)$ *is not solvable, then* $y(x)$ *is not representable by radicals.*

As we already know one can easily reduce the first statement to linear algebra (see Theorem 8 below).

We prove the second statement topologically

without using Galois theory. Vladimir Igorevich Arnold found the first topological proof of this statement.

We use another topological approach.

This first result of topological Galois gave a hint for its further development.

MONODROMY GROUP AND GALOIS GROUP

Consider the equation (1). Let

$$\Sigma \subset \mathbb{C}^N$$

be the hypersurface defined by equation

$$P_n J = 0,$$

where P_n is the leading coefficient and J is the discriminant of the equation (1).

The *monodromy group* of the equation (1) is the group of all permutations of its solutions which are induced by motions around the singular set Σ of the equation (1).

Below we discuss this definition more precisely.

At a point

$$x_0 \in \mathbb{C}^N \setminus \Sigma$$

the set

$$Y_{x_0}$$

of all germs of analytic functions satisfying equation (1) contains exactly n elements, i.e.

$$Y_{x_0} = \{y_1, \dots, y_n\}.$$

Indeed, if

$$P_n(x_0) \neq 0$$

then for

$$x = x_0$$

equation (1) has n roots counted with multiplicities.

If in addition

$$J(x_0) \neq 0$$

then all these roots are simple. By the implicit function theorem each simple root can be extended to a germ of a regular function satisfying the equation (1).

Consider a closed curve γ in

$$\mathbb{C}^N \setminus \Sigma$$

beginning and ending at the point x_0. Given a germ $y \in Y_{x_0}$ we can continue it along the loop γ to obtain another germ $y_\gamma \in Y_{x_0}$. Thus each such loop γ corresponds to a permutation

$$S_\gamma : Y_{x_0} \to Y_{x_0}$$

of the set Y_{x_0} that maps a germ $y \in Y_{x_0}$ to the germ

$$y_\gamma \in Y_{x_0}.$$

It is easy to see that the map

$$\gamma \to S_\gamma$$

defines a homomorphism from the fundamental group

$$\pi_1(\mathbb{C}^N \setminus \Sigma, x_0)$$

of the domain

$$\mathbb{C}^N \setminus \Sigma$$

with the base point x_0 to the group $S(Y_{x_0})$ of permutations. The *monodromy group* of the equation (1) is the image of the fundamental group in the group $S(Y_{x_0})$ under this homomorphism.

Remark 1. *Instead of the point* x_0 *one can choose any other point*

$$x_1 \in \mathbb{C}^N \setminus \Sigma.$$

Such a choice will not change the monodromy group up to an isomorphism.

To fix this isomorphism one can choose any curve

$$\gamma : I \to \mathbb{C}^N \setminus \Sigma$$

where I *is the segment* $0 \leq t \leq 1$ *and*

$$\gamma(0) = x_0, \quad \gamma(1) = x_1$$

and identify each germ y_{x_0} *of solution of (1) with its continuation* y_{x_1} *along* γ.

Instead of the hypersurface

$$\Sigma$$

one can choose any bigger algebraic hypersurface

$$D, \quad \Sigma \subset D \subset \mathbb{C}^N.$$

Such a choice will not change the monodromy group:

one can slightly move a curve $\gamma \in \pi_1(\mathbb{C}^N \setminus \Sigma, x_0)$ *without changing the map* S_γ *in such a way that* γ *will not intersect* D*.*