Lecture 4, September 22

CONTINUATION OF PREVIOUS LECTURE

Equations of the third degree. Suppose that the field K contains all three cubic roots of unity. On the polynomial ring $K[x_1, x_2, x_3] = V$, there is an action of the permutation group $S(3)$ of three elements. The alternating group $A(3)$, which is a cyclic group of order 3, is a normal subgroup of the group $S(3)$.

The group $A(3)$ is generated by the operator B defining the permutation x_2, x_3, x_1 of the variables x_1, x_2, x_3.

The quotient group $S(3)/A(3)$ is a cyclic group of order 2.

Let V_1 denote the invariant subalgebra of the group $A(3)$ (consisting of all polynomials that remain unchanged under all even permutations of the variables), and V_2 the algebra of symmetric polynomials.

The element x_1 has three Lagrange resolvents with respect to the generator B of the group $A(3)$:

$$R_1 = \frac{1}{3}(x_1 + x_2 + x_3),$$

$$R_{\xi_1} = \frac{1}{3}(x_1 + \xi_2 x_2 + \xi_2^2 x_3),$$

$$R_{\xi_2} = \frac{1}{3}(x_1 + \xi_1 x_2 + \xi_1^2 x_3),$$

where $\xi_1, \xi_2 = \frac{-1 \pm \sqrt{-3}}{2}$ are the cubic roots of unity different from one.

We have

$$x_1 = R_1 + R_{\xi_1} + R_{\xi_2},$$

and R_1^3, $R_{\xi_1}^3$, $R_{\xi_2}^3$ lie in the algebra V_1.

Moreover, the resolvent R_1 is a symmetric polynomial, and the polynomials $R^3_{\xi_1}$ and $R^3_{\xi_2}$ are interchanged by the action of the group $\mathbb{Z}_2 = S(3)/A(3)$ on the ring V_1.

Applying the construction used for solving quadratic equations to the polynomials $R^3_{\xi_1}$ and $R^3_{\xi_2}$, we obtain that these polynomials can be expressed through the symmetric polynomials

$$R^3_{\xi_1} + R^3_{\xi_2}, \ \ (R^3_{\xi_1} - R^3_{\xi_2})^2.$$

We finally obtain that the polynomial x_1 can be expressed through the symmetric polynomials

$$R_1 \in V_2, \ |R^3_{\xi_1} + R^3_{\xi_2} \in V_2 \ \ (R^3_{\xi_1} - R^3_{\xi_2})^2 \in V_2.$$

with the help of square and cubic root extractions and the arithmetic operations. To write down an explicit formula for the solution, it remains only to express these symmetric polynomials through the elementary symmetric polynomials.

Equations of the fourth degree. The reason for equations of the fourth degree being solvable is that the group $S(4)$ is solvable.

The group $S(4)$ is solvable because there exists a homomorphism $\pi : S(4) \to S(3)$, whose kernel is the commutative group $Kl = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

The homomorphism π can be described in the following way. **There exist exactly three ways to split a four-element set into pairs of elements.**
Every permutation of the four elements gives rise to a permutation of these splittings.

This correspondence defines the homomorphism π. The kernel Kl of this homomorphism is a normal subgroup of the group $S(4)$ consisting of four permutations:
the identity permutation and

the three permutations, each of which is a product of two disjoint transpositions.

Suppose that the field K contains all three cubic roots of unity. The group $S(4)$ acts on the polynomial ring $K[x_1, x_2, x_3, x_4] = V$. Let V_1 denote the invariant subalgebra of the normal subgroup Kl of the group $S(4)$.

Thus the polynomial ring $V = K[x_1, x_2, x_3, x_4]$ carries an action of the commutative group Kl with the invariant subalgebra V_1.

On the ring V_1, there is an action of the solvable group $S(3) = S(4)/Kl$, and the invariant subalgebra with respect to this action is the ring V_2 of symmetric polynomials.

Let A and B be operators corresponding to the permutations x_2, x_1, x_4, x_3 and x_3, x_4, x_1, x_2 of the variables x_1, x_2, x_3, x_4.

The operators A and B generate the group Kl.

The following identities hold: $A^2 = B^2 = E$.

The roots of the polynomial $T(t) = t^2 - 1$ annihilating the operators A and B are equal to $+1$, -1.

The group Kl is the sum of two copies of the group with two elements, the first copy being generated by A, and the second copy by B.

The element x_1 has four Lagrange resolvents with respect to the action of commuting operators A and B generating the group Kl:

$$R_{1,1} = \frac{1}{4}(x_1 + x_2 + x_3 + x_4),$$

$$R_{-1,1} = \frac{1}{4}(x_1 - x_2 + x_3 - x_4),$$

$$R_{1,-1} = \frac{1}{4}(x_1 + x_2 - x_3 - x_4),$$

$$R_{-1,-1} = \frac{1}{4}(x_1 - x_2 - x_3 + x_4).$$

The element x is equal to the sum of these resolvents:

$$x_1 = R_{1,1} + R_{-1,1} + R_{1,-1} + R_{-1,1},$$

the squares $R^2_{1,1}$, $R^2_{-1,1}$, $R^2_{1,-1}$, $R^2_{-1,1}$ of the Lagrange resolvents belong to the algebra V_1.

Therefore, x_1 is expressible through the elements of the algebra V_1 with the help of the arithmetic operations and square root extractions.

In turn, the elements of the algebra V_1 can be expressed through symmetric polynomials, since this algebra carries an action of the group $S(3)$ with the invariant subalgebra V_2 (see solution of cubic equations above).

Let us show that this argument provides an explicit reduction of a fourth degree equation to a cubic equation. Indeed, the resolvent $R_{1,1} = \frac{1}{4}\sigma_1$ is a symmetric polynomial.

The squares of the resolvents $R_{-1,1}$, $R_{1,-1}$ and $R_{-1,1}$ are permuted under the action of the group $S(4)$ (see the description of the homomorphism $\pi : S(4) \to S(3)$ above).

Since the elements $R^2_{-1,1}$, $R^2_{1,-1}$ and $R^2_{-1,1}$ are only being permuted, the elementary symmetric polynomials of them are invariant under the action of the group $S(4)$ and hence belong to the ring V_2. Thus the polynomials

$$b_1 = R^2_{-1,1} + R^2_{1,-1} + R^2_{-1,1},$$

$$b_2 = R^2_{-1,1}R^2_{1,-1} + R^2_{1,-1}R^2_{-1,-1} + R^2_{-1,-1}R^2_{-1,1},$$

$$b_3 = R^2_{-1,1}R^2_{1,-1}R^2_{-1,-1}$$

are symmetric polynomials of x_1, x_2, x_3 and x_4.

Therefore, b_1, b_2 and b_3 are expressible explicitly through the coefficients of the equation

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{4}$$

whose roots are x_1, x_2, x_3, x_4.

To solve equation (4), it suffices to solve the equation

$$r^3 - b_1r^2 + b_2r - b_3 = 0 \tag{5}$$

and set $x = \frac{1}{4}(-a_1 + \sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3})$, where r_1, r_2 and r_3 are the roots of equation (5).

ANOTHER REDUCTION OF A FOUR DEGREE EQUATION TO A THIRD DEGREE EQUATION

Theorem 1. *The coordinates of the intersection points of two conics $P = 0$ and $Q = 0$, where P and Q are given second degree polynomials of x and y, can be found by solving one cubic and several quadratic equations*

Indeed, every conic of the pencil

$$P + \lambda Q = 0,$$

where λ is an arbitrary parameter, passes through the points we are looking for.

For some value λ_0 of the parameter λ the conic $P + \lambda Q = 0$ splits into a pair of lines.

This value satisfies the cubic equation

$$\det(\tilde{P} + \lambda\tilde{Q}) = 0,$$

where $\tilde{P}$ and $\tilde{Q}$ are 3×3-matrices of the quadratic forms corresponding to the equations of the conics in homogeneous coordinates.

The equation for each of the lines forming the degenerate conic $P + \lambda_0 Q = 0$ can be found by solving a quadratic equation.

Indeed, the center of a degenerate conic given in affine coordinates by an equation $f(x, y) = 0$, i.e. the intersection point of the two lines forming the degenerate conic, can be found by solving the system

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

.

This is a linear system, thus a solution can be expressed as a rational function of the coefficients.

The intersection of a conic with any given line not passing through the center of the conic can be found by solving a quadratic equation. The two lines forming the degenerate conic are the lines connecting the center of the conic with the two intersection points. An equation of the line passing through two given points can be found with the help of arithmetic operations.

If the equations of the lines, into which the conic

$$P + \lambda_0 Q = 0$$

splits, are known, then to find the desired points, it remains only to solve the quadratic equations on the intersection points of the conic $P = 0$ and each of the two lines constituting the degenerate conic.

Theorem 2. *Therefore, the general equation of the fourth degree reduces to a cubic equation with the help of arithmetic operations and quadratic root extractions.*

Indeed, the roots of an equation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$

are projections to the x-axis of the intersection points of the conics

$$y = x^2 \quad \text{and} \quad a_0y^2 + a_1xy + a_2y + a_3x + a_4 = 0.$$