Free Semigroup Algebras
A survey

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$\mathcal{H}$ — separable Hilbert space.
$S_1, \dots, S_n$ — n isometries with pairwise orthogonal ranges.

$$S_i^* S_j = \delta_{ij} I \quad \text{or} \quad \sum_{i=1}^{n} S_i S_i^* \leq I$$

Free semigroup algebra:

the unital algebra $\mathfrak{S}$ generated by $S_1, \dots, S_n$
closed in the weak operator topology (WOT).

$n = 1$ is different, and we are mostly interested in $n \geq 2$.

$\underline{n=1}$. S isometry.
U_+ unilateral shift., U unitary, spectral measure $\approx \mu$.
Wold decomposition: $S \simeq U_+^{(\alpha)} \oplus U$

$$\mathfrak{S}(S) \simeq \begin{cases} L^\infty(\mu) & \text{if } \alpha = 0 \text{ and } m \not\ll \mu \\ H^\infty \oplus L^\infty(\mu_s) & \text{if } \alpha \geq 1 \text{ or } m \ll \mu. \end{cases}$$

The norm closed algebras generated by S are also of interest:

$$\mathcal{A}(S) \simeq \begin{cases} C(\sigma(U)) & \text{if } \alpha = 0 \text{ and } \sigma(U) \neq \mathbb{T} \\ A(\mathbb{D}) & \text{if } \alpha \geq 1 \text{ or } \sigma(U) = \mathbb{T}. \end{cases}$$

$$C^*(S) \simeq \begin{cases} C(\sigma(U)) & \text{if } \alpha = 0 \\ C^*(U_+) & \text{if } \alpha \geq 1. \quad \text{Toeplitz algebra} \end{cases}$$

$\underline{n \geq 2}$

$$C^*(S_1, \dots, S_n) = \begin{cases} \mathcal{O}_n & \text{if } \sum_{i=1}^n S_i S_i^* = I \quad \text{Cuntz algebra} \\ \mathcal{E}_n & \text{if } \sum_{i=1}^n S_i S_i^* < I \quad \text{Cuntz-Toeplitz algebra.} \end{cases}$$

$\mathcal{E}_n$ is a non-trivial extension of the compacts by $\mathcal{O}_n$:

$$0 \to \mathfrak{K}(H) \to \mathcal{E}_n \to \mathcal{O}_n \to 0.$$

Since $\mathcal{O}_n$ is simple and non-type I, it has a very complicated representation theory up to unitary equivalence.

$$\mathfrak{A}_n = \mathrm{Alg}(S_1, \dots, S_n) \quad \text{non-commutative disk algebra.}$$

(Popescu) $\mathfrak{A}_n$ is independent of S.

The n-shift.

Free semigroup on n letters: $\mathbb{F}_n^+$
all words w in alphabet $1, 2, \dots, n$ including $\varnothing$ as unit.

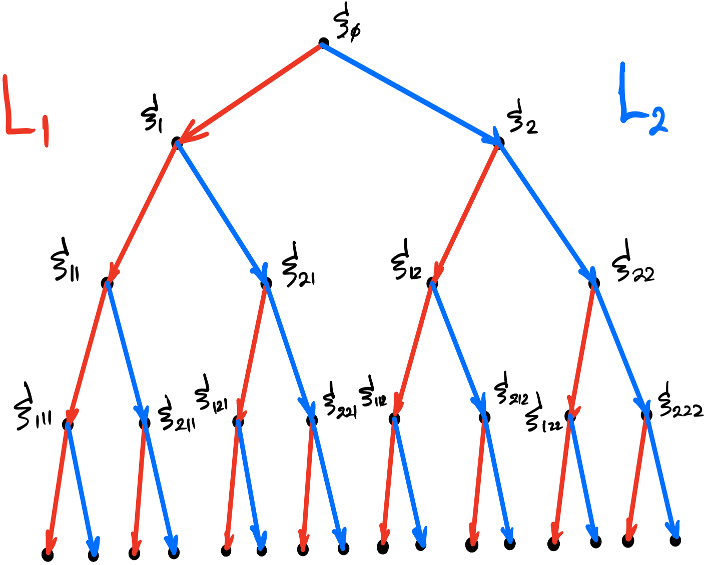
Fock space $\mathbb{F}_n^2 := \ell^2(\mathbb{F}_n^+)$. Orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$.

Left regular representation: $L_v \xi_w = \xi_{vw}$.

$$L_i^* L_j = \delta_{ij} \quad \text{and} \quad \sum_{i=1}^{n} L_i L_i^* = I - \xi_\varnothing \xi_\varnothing^*$$

Non-commutative analytic Toeplitz algebra $\mathfrak{L}_n$
is the WOT-closed algebra generated by $L_1, \dots, L_n$.
Introduced by Popescu and independently by D-Pitts.

$$\mathrm{Alg}(L_1, \dots, L_n) = \mathfrak{A}_n \quad \text{and} \quad C^*(L_1, \dots, L_n) = \mathcal{E}_n.$$


L_1
L_2
$\xi_\varnothing$
ξ_1
ξ_2
ξ_{11}
ξ_{21}
ξ_{12}
ξ_{22}
ξ_{111}
ξ_{211}
ξ_{121}
ξ_{221}
ξ_{212}
ξ_{122}
ξ_{222}

For $v \in \mathbb{F}_n^+$, $L_v \xi_w = \xi_{vw}$.

Right shift: $R_i \xi_w = \xi_{wi}$ and $R_v \xi_w = \xi_{wv^t}$, v^t reverses letters in v.
$\mathfrak{R}_n = \overline{\mathrm{Alg}(R_1, \dots, R_n)}^{\mathrm{WOT}}$ unitarily equiv. to $\mathfrak{L}_n$ via $U\xi_w = \xi_{w^t}$.

If $A \in \mathfrak{L}_n$, then $A\xi_\varnothing = \sum a_w \xi_w$.

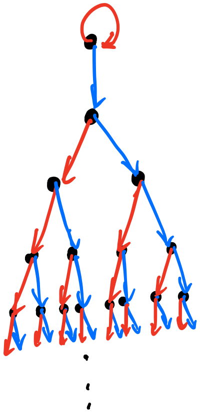
$$A\xi_v = AR_{v^t}\xi_\varnothing = R_{v^t}A\xi_\varnothing = \sum a_w \xi_{wv}.$$

Toeplitz form!. Fourier series and Cesaro means:

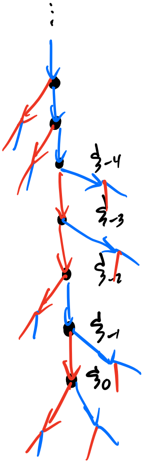
$$A \sim \sum a_w L_w = \textsc{wot}\text{–}\lim \sum_{|w|<N} \Big(1 - \frac{|w|}{N}\Big) a_w L_w.$$

Theorem $\mathfrak{L}_n' = \mathfrak{R}_n$.

Other Examples



$$\mathfrak{S} \simeq \begin{bmatrix} \mathbb{C} & 0 \\ F_n^2 & \mathfrak{L}_n \end{bmatrix}$$



$$\mathfrak{S} \simeq \mathfrak{L}_n$$

Popescu-Wold Decomposition for $S_1, \dots, S_n$.

Let $W = \big(\sum_{i=1}^n S_i S_i^* \big)^\perp H$. This is a wandering space.

$$S \simeq L^{(d)} \oplus T$$

where $d = \dim W$, $S|_{\overline{\mathfrak{S}W}} \simeq L^{(d)}$ and $T = S|_{(\mathfrak{S}W)^\perp}$ is Cuntz type.

Frahzo–Bunce–Popescu dilation: $(n = 2,\ n < \infty,\ n = \infty$ resp.)
$A = [A_1 \ \dots \ A_n]$ row contraction; i.e. $\|A\| = \big\| \sum A_i A_i^* \big\|^{1/2} \le 1$.
Dilate A to a row isometry à la Sz.-Nagy to row isometry

$$S = [S_1 \ \dots \ S_n] \quad \text{where} \quad S_i = \begin{bmatrix} A_i & 0 \\ * & * \end{bmatrix} \text{ on } H \oplus H'.$$

- If $\|A\| < 1$, then $S \simeq L^{(d)}$. In matrix case, $d \leq k$.

  (Popescu) $\mathfrak{L}_n$-functional calculus: $F \in \mathfrak{L}_n$, then

  $$F(A) = P_H F^{(d)}|_H.$$

  In modern language, F is a bounded nc function on

  $$\mathfrak{B}_n = \dot{\bigcup}_{k\geq 1} \{A \in \mathcal{M}_k^{(n)} : \|A\| < 1\}.$$

- If A_i are $k \times k$ matrices and $\sum A_i A_i^* = I$, then S is Cuntz. Finitely correlated representations (Bratteli-Jorgensen). Classifed by D-Kribs-Shpigel.

Beurling's Theorem for $\mathfrak{L}_n$

- $\mathfrak{S}_1 = \mathcal{T}(H^\infty) = \mathcal{T}(H^\infty)'$.
- (Beurling): $\operatorname{Lat} \mathcal{T}(H^\infty) = \{\operatorname{Ran} T : T \in \operatorname{Isom}(\mathcal{T}(H^\infty))\}$.
- If $R \in \mathfrak{R}_n$, then $\overline{R\mathbb{F}_n^2} \in \operatorname{Lat} \mathfrak{L}_n$.

THEOREM(Arias–Popescu, D-Pitts)

- Every invariant subspace of $\mathfrak{L}_n$ is the orthogonal direct sum of cyclic invariant subspaces.
- Cyclic invariant subspaces are $R\mathbb{F}_n^2$ for $R \in \operatorname{Isom}(\mathfrak{R}_n)$.

COROLLARY: $\mathfrak{L}_n$ is reflexive.

Inner-outer factorization.

$A \in \mathfrak{L}_n$, $M = \overline{A\mathbb{F}_n^2} \in \operatorname{Lat} \mathfrak{R}_n$, $A\xi_\varnothing$ is cyclic for M.
So $\exists S \in \operatorname{Isom}(\mathfrak{L}_n)$ (inner), $S\mathbb{F}_n^2 = M$.
Thus $A = SB$ where $B = S^* A \in \mathfrak{R}_n' = \mathfrak{L}_n$.
$\overline{B\mathbb{F}_n^2} = \mathbb{F}_n^2$ (outer).

Eigenvectors

$\lambda \in \mathbb{B}_n = \{\lambda \in \mathbb{C}^n : \|\lambda\|_2 < 1\}$. Unit vectors:

$$\nu_\lambda = (1 - \|\lambda\|^2)^{1/2} (I - \sum_{i=1}^{n} \overline{\lambda_i} L_i)^{-1} \xi_\varnothing .$$

- The joint eigenvectors of L_i, $1 \le i \le n$, are $\{\nu_\lambda : \lambda \in \mathbb{B}_n\}$ and

$$L_i^* \nu_\lambda = \overline{\lambda_i} \nu_\lambda . \quad \text{for} \quad 1 \le i \le n.$$

- The function $\hat{A}(\lambda) = \varphi_\lambda(A) := \langle A\nu_\lambda, \nu_\lambda \rangle$ is a contractive homomorphism into $H^\infty(\mathbb{B}_n)$.
- $\{\varphi_\lambda : \lambda \in \mathbb{B}_n\}$ are the WOT-continuous characters of $\mathfrak{L}_n$.
- $\{\nu_\lambda\}$ span symmetric Fock space or Drury-Arveson space H^2_n.
- $\bigcap_\lambda \ker \varphi_\lambda = \overline{\{AB - BA : A, B \in \mathfrak{L}_n\}}^{\text{WOT}} =: \mathcal{C}$.

THEOREM (DP) $\quad \mathfrak{L}_n / \mathcal{C} \simeq \text{Mult}(H^2_n)$.

Automorphisms

Let $\Theta \in \mathrm{Aut}(\mathfrak{L}_n)$. $\tau_\Theta(\lambda) := \big(\varphi_\lambda \Theta^{-1}(L_1), \dots, \varphi_\lambda \Theta^{-1}(L_n)\big)$ is holomorphic from $\mathbb{B}_n$ into itself. And

$$\tau_{\Theta_1}\tau_{\Theta_2} = \tau_{\Theta_1\Theta_2} \quad \text{and} \quad \tau_{\Theta^{-1}} = \tau_\Theta^{-1}.$$

Thus $\tau_\Theta \in \mathrm{Aut}(\mathbb{B}_n)$ is a conformal automorphism of $\mathbb{B}_n$.

THEOREM(Voiculescu): $\exists u$ from $\mathrm{Aut}(\mathbb{B}_n)$ to $\mathcal{U}(\mathbb{F}_n^2)$ so that $\mathrm{Ad}(u(\theta))(A) = UAU^* \in \mathrm{Aut}(\mathcal{E}_n)$ and $\mathrm{Ad}(u(\theta))(\mathfrak{A}_n) = \mathfrak{A}_n$.

- $\Theta_\theta = \mathrm{Ad}(u(\theta))(\mathfrak{L}_n) = \mathfrak{L}_n$ onto itself.
- Θ_θ is completely isometric and WOT continuous.
- $\tau : \mathrm{Aut}(\mathfrak{L}_n) \to \mathrm{Aut}(\mathbb{B}_n)$ is surjective.

THEOREM (DP)

- $\Theta \in \mathrm{Aut}(\mathfrak{L}_n)$ are norm and WOT-continuous.
- $0 \longrightarrow \text{q-Inn}(\mathfrak{L}_n) \longrightarrow \mathrm{Aut}(\mathfrak{L}_n) \xrightarrow{\tau} \mathrm{Aut}(\mathbb{B}_n) \longrightarrow 0$
  where $\text{q-Inn} = \ker \tau$ — autos trivial mod $\mathcal{C}$.
- sequence splits via Voiculescu's map onto unitarily implemented autos.

QUESTION: Is $\text{q-Inn}(\mathfrak{L}_n)$ non-trivial?

Structure Theory of Free Semigroup Algebras

Let the generators for $\mathcal{E}_n$ be $\mathfrak{s}_1, \dots, \mathfrak{s}_n$ and $\sigma \in \mathrm{Rep}(\mathcal{E}_n)$.
Define $\mathfrak{S}_\sigma = \overline{\mathrm{Alg}\{S_i = \sigma(\mathfrak{s}_i)\}}^{\mathrm{WOT}}$ and $\mathfrak{W}_\sigma = \mathrm{C}^*(\{S_i\})''$.
Left regular repn. is λ; and $\mathfrak{S}_\lambda = \mathfrak{L}_n$.

- Say that σ is type L or analytic if $\mathfrak{S}_\sigma$ is isomorphic to $\mathfrak{L}_n$.
- Say that σ is von Neumann type if $\mathfrak{S}_\sigma = \mathfrak{W}_\sigma$.
- Say that σ is dilation type if $\mathfrak{S}_\sigma$ is there is a co-invariant, cyclic subspace $\mathcal{V}$ such that $S|_\mathcal{V}$ is c.n.i.

$\mathfrak{S}_0 = \langle S_1, \dots, S_n \rangle$ is the WOT-closed ideal generated by $\{S_i\}$.

- $\mathfrak{S}$ is a von Neumann algebra $\iff \mathfrak{S}_0 = \mathfrak{S}$.
- $\mathfrak{J} := \bigcap_{k \ge 1} \mathfrak{S}_0^k$ is a left ideal of $\mathfrak{W}$.
  So there is a projection $P \in \mathfrak{S}_\sigma$ so that $\mathfrak{J} = \mathfrak{W}P$.

THEOREM (D-Katsoulis-Pitts) Structure Theorem

- Type L algebras are completely isometrically isomorphic and weak-$*$ homeomorphic to $\mathfrak{L}_n$.
- $P\mathcal{H}$ is coinvariant and if $P \neq I$, $\mathfrak{S}_\sigma|_{P^\perp\mathcal{H}}$ is type L.

$$\mathfrak{S}_\sigma = \begin{bmatrix} P\mathfrak{W}_\sigma P & 0 \\ P^\perp\mathfrak{W}_\sigma P & \mathfrak{S}_\sigma|_{P^\perp\mathcal{H}} \end{bmatrix}$$

THEOREM (Read) $\mathcal{B}(H)$ is a free semigroup algebra.

QUESTION Which von Neumann algebras are FSG algebras?

- Must be infinite and injective.

Absolute Continuity

$$A(\mathbb{D})^* \simeq H^\infty_* \oplus_1 M_s(\mathbb{T}).$$

$$A(\mathbb{B}_n)^* \simeq H^\infty(\mathbb{B}_n)_* \oplus_1 \mathrm{TS}(\partial\mathbb{B}_n).$$

DEFINITION

- $\varphi \in \mathfrak{A}_n^*$ is absolutely continuous if $\exists \zeta, \eta \in \mathbb{F}_n^2$ such that
  $$\varphi(A) = \langle \lambda(A)\zeta, \eta \rangle.$$
- $\xi \in H_\sigma$ is absolutely continuous if $\varphi(A) = \langle \sigma(A)\xi, \xi \rangle$ is a.c.
- $\mathfrak{S}$ is absolutely continuous if every $\xi \in H$ is a.c.
- A vector ξ is wandering if $\{S_w\xi : w \in \mathbb{F}_n^+\}$ is orthonormal.

THEOREM (D-Li-P, D-Yang)

- $\mathfrak{S}_\sigma$ is a.c. $\iff \sigma \oplus \lambda$ is type L $\iff \sigma^{(\infty)}$ is type L.
- $\mathfrak{S}$ type L $\iff \exists k$ $\mathfrak{S}_{\sigma^{(k)}}$ is spanned by wandering vectors.
- $\mathfrak{S}$ a.c. and a wandering vector $\iff$ analytic and wandering vectors span H.

The final big step

THEOREM (Kennedy) Every a.c. FSG has a wandering vector.
COROLLARY $\mathfrak{S}$ a.c. implies analytic and spanned by wandering vectors.

Idea: modify dual algebra techniques of Brown, Chevreau, Pearcy, Bercovici, Foiaș, etc.

$\varphi_0 : \mathfrak{S} \to \mathfrak{S}/\mathfrak{S}_0 \simeq \mathbb{C}$ is weak-$*$ cnts..
Want $\zeta, \eta \in H$ so that $\varphi_0(A) = \langle A\zeta, \eta\rangle$.
Then $\eta \perp \overline{\mathfrak{S}_0\zeta}$.
$\xi \in \overline{\mathfrak{S}\zeta} \ominus \overline{\mathfrak{S}_0\zeta}$ is wandering.

COROLLARY Lebesgue-von Neumann-Wold decomposition
S row isometry, then $S \simeq L^{(\alpha)} \oplus S_a \oplus S_{vN} \oplus S_d$, where
S_a is a.c., Cuntz, S_{vN} is v.N. type, and S_d is dilation type.

Hyperreflexivity

If $\mathfrak{A}$ is reflexive, $A \in \mathfrak{A}$, $T \in \mathcal{B}(\mathcal{H})$ and $M \in \operatorname{Lat} \mathfrak{A}$, then

$$\|P_M^\perp T P_M\| = \|P_M^\perp (T - A) P_M\|.$$

therefore

$$\beta_{\mathfrak{A}}(T) := \sup_{M \in \operatorname{Lat}(\mathfrak{A})} \|P_M^\perp T P_M\| \leq \operatorname{dist}(T, \mathfrak{A}).$$

An operator algebra $\mathfrak{A}$ is hyper-reflexive if

$$\operatorname{dist}(T, \mathfrak{A}) \leq C \beta_{\mathfrak{A}}(T) \text{ for all } T \in \mathcal{B}(\mathcal{H})$$

THEOREM (D) $\mathcal{T}(H^\infty)$ is hyper-reflexive with $C \leq 19$.

Theorems

- (DP) $\mathfrak{L}_n$ is hyper-reflexive with $C \leq 51$.
- (Bercovici) $\mathfrak{L}_n$ is hyper-reflexive with $C \leq 3$.
- (D-Li-P) Every analytic FSG alg. with wandering vector is hyper-reflexive with $C \leq 59$.
- (Kennedy) Every analytic FSG alg. is hyper-reflexive with $C \leq 3$.
- (Fuller-Kennedy) Every FSG alg. is hyper-reflexive with $C \leq 6$.

Singular Functionals

Say $\varphi \in \mathfrak{A}_n^*$ is singular if $\|\varphi|_{(\mathfrak{A}_n^0)^k}\| = \|\varphi\|$ for all $k \geq 1$.
One can show that $\mathfrak{A}_n^{**}$ is a FSG algebra, and φ is weak-$*$ cnts on it and annihilates the type L part.

Proposition (D-Li-P)

$\varphi \in \mathfrak{A}_n^*$ decomposes uniquely: $\varphi = \varphi_a + \varphi_s$, $\varphi_a \in \mathfrak{L}_{n*}$, φ_s singular.

$$\|\varphi\| \leq \|\varphi_a\| + \|\varphi_s\| \leq \sqrt{2}\|\varphi\|.$$

$$\mathfrak{A}_n^* = \mathfrak{L}_{n*} \dot{+} (\mathfrak{A}_n^*)_s.$$

Kennedy and Yang work in $\mathfrak{L}_n^*$ instead, which works the same.

Theorem (Kennedy-Yang) An F & M Riesz Theorem

If $\varphi \in \mathfrak{L}_n^*$ and J is a WOT-closed ideal such that $\varphi \perp J$, then $\varphi_a \perp J$ and $\varphi_s \perp J$.

Corollary This decomposition and F&M Riesz result pass to quotients by WOT-closed ideals. In particular, it applies to multipliers of complete NP kernels.

Recent Developments

- Think of $\mathbb{F}_n^2$ as $H^2_{nc} = \{f(z) = \sum_{w \in \mathbb{F}_n^+} a_w z^w\}$ in non-commuting variables.
- (Ball-Marx-Vinnikov) H^2_{nc} is an nc RKHS on the nc ball $\mathfrak{B}_n = \dot{\bigcup}_{k \ge 1} \{A \in \mathcal{M}_k^{(n)} : \|A\| < 1\}$.
- (Popescu, Salomon-Shalit-Shamovich) $\mathfrak{L}_n = \mathrm{Mult}(H^2_{nc})$.
- (Jury-Martin) Fatou's theorem. Alexandrov-Clark theory.
- (Jury-Martin) factorization in $H \odot H$ for CNP spaces.
- (Jury-Martin-Shamovich) Every inner function factors as a Blaschke factor and a singular factor.

The end.
Thanks for your attention.