# Von Neumann conditional expectations and noncommutative representing measures

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(Joint with L. E. Labuschagne)

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## Abstract

We begin by reviewing the theory of normal conditional expectations on von Neumann algebras. Second, we describe several new contributions to this theory, such as establishing a correspondence between normal conditional expectations onto a subalgebra, and 'noncommutative weight' functions affiliated with a centralizer in the algebra. Third, inspired by Arveson's noncommutative Hardy space theory, we introduce a new kind of noncommutative representing measures of 'non-commutative characters' on noncommutative function algebras. Finally, we consider and give successively more general solutions to the problem of the 'existence of representing measures' in the weak* case. That is, when do there exist weak* continuous positive extensions of non-commutative characters on such noncommutative function algebras?

(Joint work with L. E. Labuschagne. To appear AiM)

## Part I. Introduction

Classical function algebra: subalgebra $A$ of commutative $C^*$-algebra $C(K)$ or $L^\infty(K, \mu)$

- In the classical books on the subject (e.g. [Gamelin]), about the first topic considered is the maximal ideal space $M_A$, or equivalently the set of (nontrivial, scalar valued) homomorphisms on $A$, a.k.a. the characters on $A$

- For subalgebra $A \hookrightarrow C(K)$ characters immediately arise from point evaluations at points in $K$

- One goes on to show that the Gelfand transform is an isometric homomorphism $A \hookrightarrow C(M_A)$; the spectrum of $f \in A$ is $\{\theta(f) : f \in M_A\}$

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- Then one discusses Choquet and Shilov boundary, peak sets and peak points, Dirichlet and logmodular algebras, generalized Hardy algebras and Hardy spaces (generalized analytic functions), Gleason parts, etc

A representing measure for a character $\theta$ of $A \hookrightarrow C(K)$ is a positive measure $\mu$ on $K$ such that

$$\theta(f) = \int_K f \, d\mu, \qquad f \in A.$$

- In functional analytic terms a representing measure is a (positive) extension of a character to a state on the containing $C^*$-algebra $C(K)$ (note representing measures are probability measures)

## Generalized Hardy spaces

• Let $\mu$ be a representing measure for a character $\theta$ of $A \hookrightarrow C(K)$. Define $H^\infty(\mu)$ to be the weak* closure of $A$ in $L^\infty(\mu)$, and similarly for $p < \infty$ define $H^p(\mu)$ as the closure of $A$ in $L^p(\mu)$

Call this $\theta$ the primary character. E.g. if $A$ is the disk algebra or $H^\infty$ of the disk, then $\theta(f) = f(0)$, and the important 'representing measure' is $\mu(f) = \int_{\mathbb{T}} f \, dm$, Lebesgue integration on the circle, which is a state on $C(\mathbb{T})$ and is a weak* conts state on $L^\infty(\mathbb{T})$.

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• If $A$ is a Dirichlet and logmodular algebra, or more generally a weak* Dirichlet algebra then these Hardy spaces behave very similarly to the classical Hardy spaces of the disk. Get F & M Riesz theorem, Beurling's theorem, Jensen and Szego, Gleason-Whitney theorem, inner-outer factorization, etc. (Due to Hoffman, Helson-Lowdenslager, and many others.)

Summary: At the heart of the 50-60's theory of generalized Hardy spaces, is a weak* continuous character of a function algebra, and the weak* continuous positive state extension corresponding to its important representing measure.

More generally, the classical Hoffman-Rossi theorem proves the existence in general of 'normal' or 'absolutely continuous' representing measures':

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More generally, the classical Hoffman-Rossi theorem proves the existence in general of 'normal' or 'absolutely continuous' representing measures':

Theorem (Hoffman-Rossi) Weak* continuous characters of a function algebra $A \subset M = L^\infty(X, \mu)$ have weak* continuous positive state extensions to $M$. Such extensions correspond to representing measures on $X$ which are absolutely continuous with respect to $\mu$.

(Comment: Should be related to the Henkin measures in Raphael's awesome talk)

We display a new swift proof (that uses an ancient idea of Sarason/Konig for a related result):

**Theorem** (Hoffman-Rossi) Weak* continuous characters of function algebra $A \subset M = L^\infty(X,\mu)$ have weak* continuous positive state extensions

**Proof.** By Banach space duality there exists $r \in L^1$ such that $\varphi(x) = \int xr$ for $x \in A$. Write $r = ab$ for $a, b \in L^2$. Let $J = \operatorname{Ker}\varphi$, and let $E$ (resp. $F$) be the closure in the $L^2$ norm of $Aa$ (resp. $Ja$). For $f \in J$ we have

$$\|a - fa\|_2^2 = \int (a^*|1-f|^2 a) = \int |(1-f)a|^2 \geq \frac{1}{C^2},$$

for a constant $C > 0$, since

$$1 = \varphi(1-f) = \int (1-f)r = \int b(1-f)a \leq C \int |(1-f)a|^2)^{\frac{1}{2}}.$$

It follows that $a \notin F$, so $E \neq F$. Choosing a unit vector $h \in E \ominus F$,

$$\int f|h|^2 = \langle hf, h\rangle = 0, \qquad f \in J,$$

since $hf \in EJ \subset F$. Since $A = J + \mathbb{C}1$ it follows that $\int(\cdot|h|^2)$ is a normal state on $M$ extending $\varphi$.

## Arvesons noncommutative Hardy spaces

At the end of the 60's Arveson generalized the then-new theory of generalized Hardy spaces above, and its 'weak* continuous character' approach, to a von Neumann algebraic setting.

Arveson's approach to noncommutative analyticity/ generalized analytic functions/noncommutative Hardy spaces $H^p(A)$

Thinking about classical setting above, with $H^\infty(\mu)$ a weak* closed subalgebra $A$ in $L^\infty(\mu)$, and $H^p(\mu)$ as the closure of $A$ in $L^p(\mu)$ for $p < \infty$, ...

Arveson setting: a (very special) weak* closed subalgebra $A$ of a von Neumann algebra $M$, and the 'primary character' replaced by a weak* continuous homomorphism $\Phi : A \to D = A \cap A^*$ (what we call a $D$-character below)

With this in mind, we define (absolutely conts) NC representing measure: normal conditional expectation $\Psi : M \to D$ extending $\Phi$

Simplest example: upper triangular matrices $A$, which lives between the two vNa's $M_n$ and $D_n$. The canonical 'expectation' $E : M_n \to D_n$ is our trace preserving nc representing measure extending the canonical $D$-character on $A$

Conditional expectation a (usually normal, that is, weak* continuous) unital positive (or equivalently, contractive) idempotent $D$-module map $E : M \to D$ from a unital $C^*$-algebra onto its unital $C^*$-subalgebra $D$

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- We also make several new contributions to the theory of conditional expectations of von Neumann algebras

- We now describe this theory, and its motivation from probability theory.

- If you are only interested in noncommutative representing measures, and not von Neumann algebra theory, feel free to zone out a little in the next section: but remember its forthcoming main point: conditional expectations on von Neumann algebras are not 'monolithic', they can be classified/parametrized/are responsive to the tools and treatment of analysis

## Part II. Von Neumann algebra conditional expectations

### Motivation from probability theory:

If $(K, \mathcal{A}, \mu)$ is a probability measure space, then von Neumann subalgebras $D$ of $M = L^\infty(K, \mathcal{A}, \mu)$ correspond (up to null sets) to 'essentially order-closed' sub-$\sigma$-algebras $\mathcal{B}$ of $\mathcal{A}$

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Also $E$ is normal and faithful (that is, $E(f) = 0$ implies $f = 0$ if $f \in M_+$).

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This unique conditional expectation will be called the weight-preserving conditional expectation. If we write it as $\mathbb{E}_\mu$ then one may show that any other normal conditional expectation $E : M \to D$ is given by:

$$E(x) = \mathbb{E}_\mu(hx)$$

for a density function $h \in L^1(K, \mathcal{A}, \mu)_+$ with $\mathbb{E}_\mu(h) = 1$. Such $h$ is sometimes called a *weight function*.

Takeaway: One may study/classify all normal conditional expectations in terms of weight functions (and the one fixed 'probabilistic conditional expectation')–so can use tools and treatment of analysis

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One final thing can be said. This any other normal conditional expectation $E : M \to D$, assuming that it is faithful, is the weight-preserving conditional expectation for a certain probability measure (which in retrospect is the obvious one induced by $\mu$ and $E$).

We show that a similar structure as in the last slides pertains in the non-commutative case

## Von Neumann algebras and noncommutative integration theory:

(Basic text on noncommutative integration theory: Takesaki volume 2)

Weight: A $[0, \infty]$-valued 'linear' functional $\omega$ on $M_+$. E.g. The integral on $L^\infty(X, \mu)_+$.

The weight is semifinite if $\exists\, x_t \nearrow 1$ with $\omega(x_t) < \infty$

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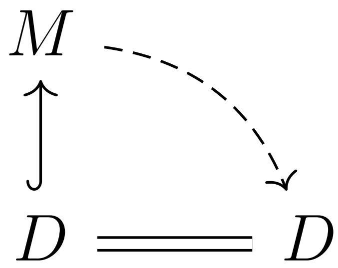
Von Neumann algebras are regarded as noncommutative $L^\infty$-spaces. Fixing a faithful normal semifinite (fns) weight $\omega$ on $M$, we view the pair $(M, \omega)$ as a noncommutative measure space.

For a von Neumann algebra inclusion $D \subset M$, the existence of a normal (weak* continuous) conditional expectation $E : M \to D$ is a rich subject, and is often a very tricky matter. The best result is as follows:

**Theorem** (Takesaki) If $\omega$ is a fns weight on $M_+$ then there exists a (necessarily unique) weight-preserving normal conditional expectation $E : M \to D$ if and only if $\omega$ restricts to a semifinite weight on $D$ and the modular group of $\omega$ leaves $D$ invariant



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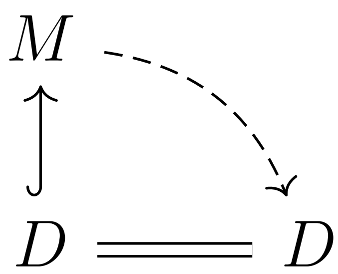

(So we should expect to see such conditions in the sequel.)

A special case: if $\omega$ restricts to a semifinite trace on $D$ then there exists an $\omega$-preserving normal conditional expectation $E : M \to D$ if and only if $D$ is contained in the centralizer of $\omega$. If in addition $\omega$ is a normal state on $M$, then this centralizer condition is saying that

$$\omega(dx) = \omega(xd), \qquad x \in M, d \in D.$$

Some complementary results: We prove some complementary results in the case that $\omega$ is not faithful on all of $M$. Some of these results are also local in character, by which we mean we relax conditions like $\omega$-preserving to a local version: $\omega = \omega \circ E$ on $pMp$ for every projection $p \in D$ with $\omega(p) < \infty$. Get e.g. 'local' weight variants but with the following flavour:

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Theorem. Let $\omega$ be a normal state on a von Neumann algebra $M$ which is faithful and tracial on a von Neumann subalgebra $\mathcal{D}$. Then

$$\omega(dx) = \omega(xd), \qquad d \in \mathcal{D}, x \in M,$$

if and only if there is a (unique) normal conditional expectation $E$ from $M$ onto $\mathcal{D}$ which preserves $\omega$.

Theorem Suppose that $D$ is a von Neumann subalgebra of a von Neumann algebra $M$. The following are equivalent:

(1) $D$ is semifinite (or equivalently, has a faithful normal semifinite trace $\tau$) and there exists a normal conditional expectation (resp. faithful normal conditional expectation) $E : M \to \mathcal{D}$.

(2) $M$ has a normal (resp. faithful normal) weight $\omega$ with $D$ 'in its centralizer' such that $\omega$ is semifinite and faithful on $D$.

(3) $M$ has a normal (resp. faithful normal) weight $\omega$ with $D$ locally $\omega$-central, such that $\omega$ is a semifinite faithful trace on $D$.

Moreover if $(2)$ holds then we can choose $E$ in $(1)$ to be $\omega$-preserving. Such an $\omega$-preserving normal conditional expectation onto $D$ is unique. Also $E \mapsto \tau \circ E$ is a bijective correspondence between the items in $(1)$ and $(2)$. This correspondence is also a bijection with the $\omega$ in $(3)$ whose support projections commute with $D$.

(In partic., we should expect to see centralizer conditions in the sequel.)

**Theorem** Let $D$ be a von Neumann subalgebra of $M$, and $\nu$ a faithful normal weight on $M$, which is semifinite on $D$, and satisfies $\sigma_t^\nu(D) \subset D$ for all $t \in \mathbb{R}$. There are bijective correspondences between:

(a) Normal conditional expectations $E$ onto $D$ which 'commute with' $(\sigma_t^\nu)$,

(b) (Weight function:) densities $h \, \eta \, M_\nu^+$ which commute with $D$, and which satisfy $E_D(h) = 1$ (where the action of $E_D$ on $h$ is by the normal extension of $E_D$ to Haagerup's extended positive part of $M$),

(c) extensions of $\nu_{|D}$ to normal semifinite weights $\rho$ on all of $M$ which 'commute with' $\nu$, whose support projection $z$ satisfies a natural property.

With respect to the correspondences above we also have the formulae

$$\rho = \nu \circ E = \nu(h^{\frac{1}{2}} \, x \, h^{\frac{1}{2}}), \qquad h = \frac{d\rho}{d\nu}, \qquad E(x) = E_D(h^{\frac{1}{2}} \, x \, h^{\frac{1}{2}}).$$

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(In the most general settings one has, as is usual in noncommutative integration, to be a bit careful with the definition of $h^{\frac{1}{2}}\, x\, h^{\frac{1}{2}}$ above.)

Takeaway: Again, one may study/classify all normal conditional expectations in terms of weight functions (and the one fixed 'probabilistic conditional expectation')

Again, one final thing can be said. Any such normal conditional expectation $E : M \to D$, assuming that it is faithful, is the weight-preserving conditional expectation for a certain weight (which in retrospect is $\nu \circ E$).

We now turn to the special case that semifinite $D$ is contained in the centralizer of the weight:

Corollary Let $D$ be a von Neumann subalgebra of $M$, and $\nu$ a faithful normal weight on $M$, which is semifinite on $D$, and contains $D$ in its centralizer. There are bijective correspondences between:

(a) Normal conditional expectations $E$ onto $D$ which 'commute with' $(\sigma_t^\nu)$,

(b) (Weight function:) densities $h\,\eta\,M_\nu^+$ which commute with $D$, and which satisfy $E_D(h) = 1$ as before,

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**Corollary** Let $D$ be a von Neumann subalgebra of $M$, and $\tau$ a faithful normal tracial state on $M$. There is a bijective correspondence between the following objects:

(a) Normal conditional expectations onto $D$,

(b) densities $h \in L^1(M)_+$ which commute with $D$ and satisfy $E_D(h) = I$,

(c) extensions of $\tau_{|D}$ to normal states $\rho$ on all of $M$ which have $D$ in their centralizer.

The normal conditional expectation $E$ in $(\mathrm{a})$ and $h$ in $(\mathrm{b})$ are related by the formula

$$E(x) = E_D(h^{\frac{1}{2}} x h^{\frac{1}{2}}), \qquad x \in M_+.$$

Also $h$ in $(\mathrm{b})$ is the Radon-Nikodym derivative $\frac{d\rho}{d\tau}$ for $\rho$ as in $(\mathrm{c})$.

## Part III. Noncommutative weak* representing measures

We propose a new merger of the two fields briefly surveyed in Parts I and II:

Noncommutative function algebra (a.k.a. operator algebra): $A \subset C$ subalgebra of $C^*$-algebra $C$, assume $1_C \in A$.

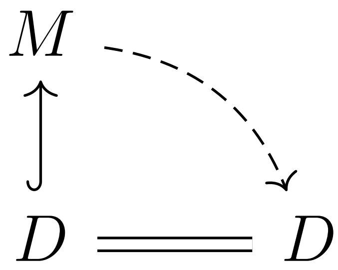
- However in this setting scalar valued characters are usually not so useful.

Reminder from function theory:

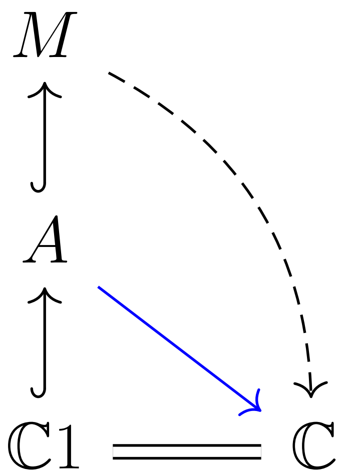
Theorem (Hoffman-Rossi) Weak* continuous characters of a function algebra $A \subset M = L^\infty(X, \mu)$ have weak* continuous positive state extensions to $M$.

To see how to merge the theory of von Neumann algebra conditional expectations and the function algebraic representing measures of characters, look at their diagrams:

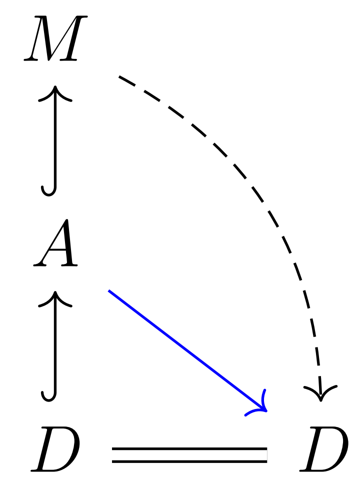
Picture of existence of conditional expectation:




Classical picture of representing measure/Hoffman-Rossi theorem:



BL 21: Merging of the theory of von Neumann algebra conditional expectations, and the function algebraic representing measures of characters:




(Case $A = D$ is the case of existence of vNa conditional expectations)

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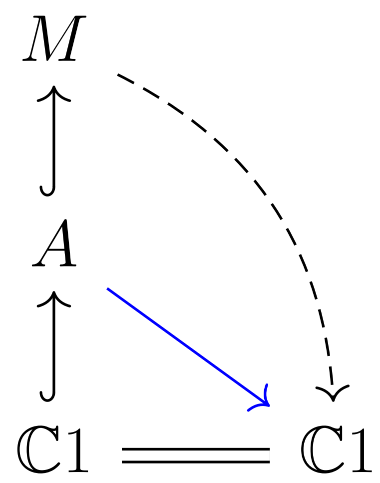
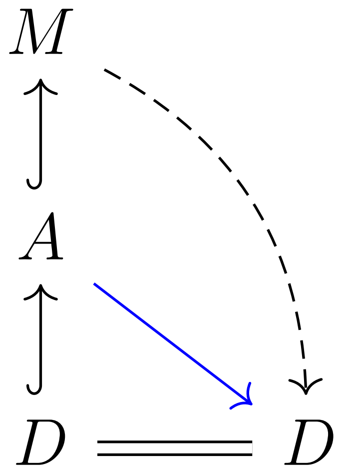


Here we have an inclusion $D \subset A \subset M$, where $D, M$ are von Neumann algebras, and $A$ is a subalgebra of $M$.

A $D$-character is a unital (completely) contractive homomorphism $\Phi : A \to D$ which is also a $D$-bimodule map. (E.g. Arveson example.)



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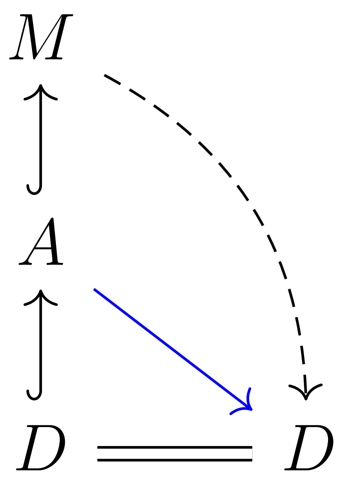
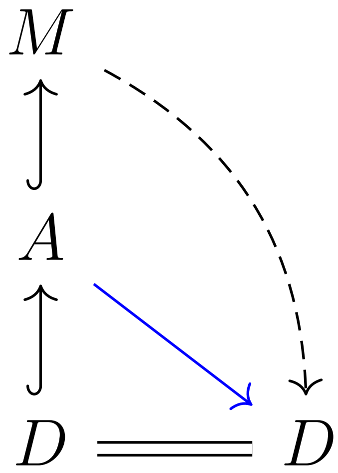



Do noncommutative representing measures for $D$-characters exist?

(Yes! – under successively more general conditions, going towards the same conditions that von Neumann algebra conditional expectations exist. The limits of our knowledge seem to suggest new directions of study in the latter field.)

Why this result should be a little unbelievable:



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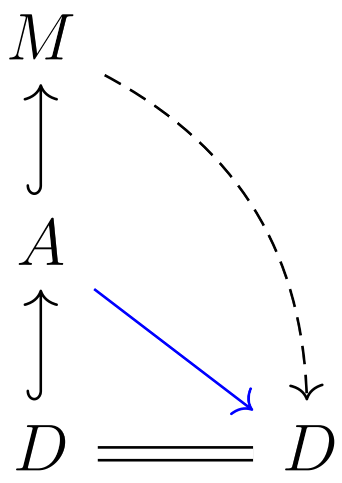


Here all maps are weak* continuous and (completely) contractive, so in particular one is obtaining a Hahn-Banach type extension theorem giving a weak* continuous contractive (and positive) extension. But it is well known that 1) there is no general weak* continuous Hahn-Banach theorem, 2) there is not even any norm continuous such Hahn-Banach theorem unless $D$ is 'injective', which we are not assuming.

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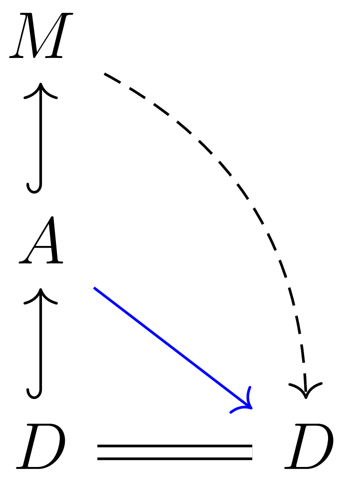


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There is some magic going on... deriving from the ancient idea of Sarason in the commutative case.

## Existence of normal representing measures:

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- In the third stage we use some technology developed in our paper for conditional expectations to prove the main result–namely, such states with $D$ in its centralizer possess associated $D$-valued normal conditional expectations.

- From the second stage: a purely von Neumann algebraic averaging technique that allows averaging of a normal state on a $\sigma$-finite von Neumann algebra $M$ over a von Neumann subalgebra $D$ of the centralizer of the ambient fns, to a obtain a normal state $\rho$ on $M$ with the additional property

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Theorem Let $\omega$ be a faithful normal state on a von Neumann algebra $M$, and let $D$ be a von Neumann subalgebra of $M_\omega$. Let $\mathcal{U}$ be the unitary group of $D$, $\psi$ a normal state extending $\omega_{|D}$, and let $K_\psi$ be the norm closed convex hull of

$$\{u^* \psi u : u \in \mathcal{U}\}$$

in $M_*$. Then $K_\psi$ contains a normal state $\rho$ extending $\omega_{|D}$ such that

$$\rho(dx) = \rho(xd), \qquad d \in \mathcal{D}, x \in M.$$

Indeed we may take $\rho$ to be $\psi \circ E_{D' \cap M}$ where $E_{D' \cap M}$ is the unique $\omega$-preserving conditional expectation onto $D' \cap M$.

## Existence of normal representing measures:

Theorem Consider the inclusions $D \subset A \subset M$, where $M$ is a von Neumann algebra with normal state $\nu$ which is faithful on von Neumann subalgebra $D$, with $D$ in the centralizer of $\nu$, and $A$ is a subalgebra of $M$. Let $\Phi : A \to D$ be a homomorphism extending the identity map on $D$).

(1) Assume that $\nu \circ \Phi$ extends to a normal state $\rho$ on $M$ with $D$ in its centralizer. Then there is a unique $\rho$-preserving normal conditional expectation $M \to D$, and this expectation extends $\Phi$.

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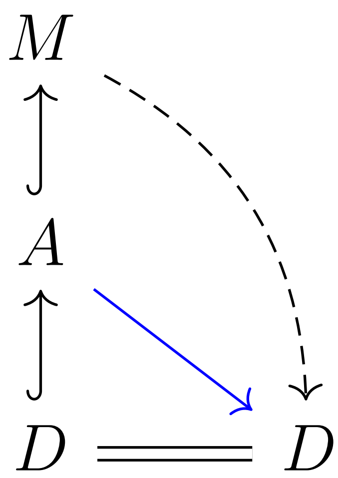
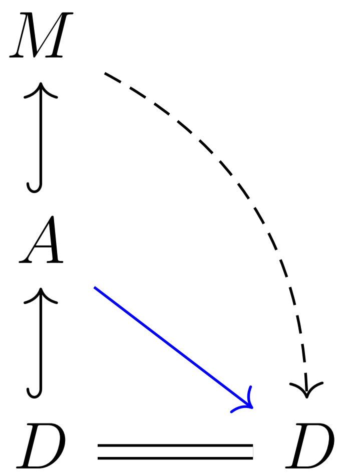
(2) Assume that $\nu$ is faithful and $\nu \circ \Phi$ extends to a normal state on $M$. Then $\nu \circ \Phi$ extends to a normal state $\rho$ on $M$ with $D$ in its centralizer, and there is a normal $\rho$-preserving conditional expectation $M \to D$ extending $\Phi$.

**Theorem ('Finite' vNa case)** Consider inclusions $D \subset A \subset M$ as above ($A$ is a weak* closed subalgebra of vNa $M$). Assume that $M$ has faithful normal tracial state, and let $\Phi : A \to D$ be a weak* continuous $D$-character. Then there exists a normal conditional expectation $\Psi : M \to D$ extending $\Phi$.



Theorems Many more general ‘existence of representing measure’ results for $D$-characters.



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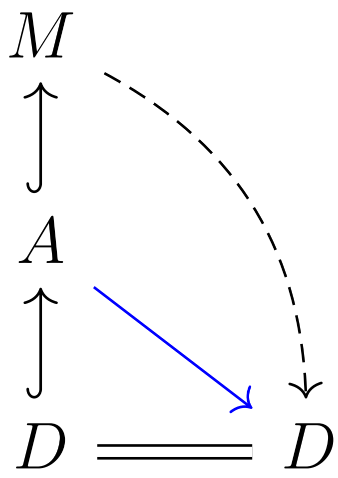


• So, for example, if $M$ is commutative, no extra conditions are needed. This is a new result in the classical case:

Theorem Consider inclusions $D \subset \mathcal{A} \subset M$, where $M$ is a commutative von Neumann algebra, $\mathcal{A}$ is a weak* closed subalgebra of $M$, and $D$ is a von Neumann subalgebra containing the unit of $M$. Let $\Phi : \mathcal{A} \to \mathcal{D}$ be a weak* continuous $D$-character. Then there is a unique $\rho$-preserving normal conditional expectation $M \to \mathcal{D}$ extending $\Phi$.

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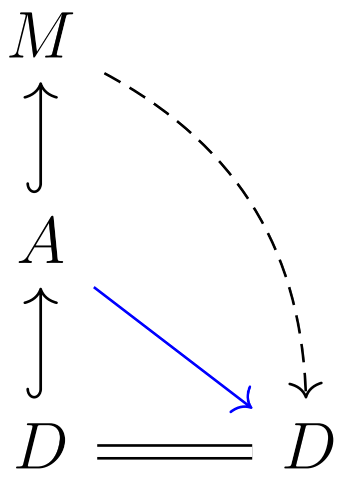


- If $D$ is atomic, no extra conditions needed [B-Flores-Zimmer]:

Theorem Consider inclusions $D \subset \mathcal{A} \subset M$, where $M$ is any von Neumann algebra, $\mathcal{A}$ is a weak* closed subalgebra of $M$, and $D$ is an atomic von Neumann subalgebra containing the unit of $M$. Let $\Phi : \mathcal{A} \to \mathcal{D}$ be a weak* continuous $D$-character. Then there is a unique $\rho$-preserving normal conditional expectation $M \to \mathcal{D}$ extending $\Phi$.

**Theorem** Same theorem (won't state, you must be getting bored), but now $M$ is $\sigma$-finite, and $D$ contained in the centralizer.

The proof is quite technical, involving the technology of Haagerup Lp-spaces.

**General von Neumann algebra case:** Results not quite as beautiful, and open questions start to appear...

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**General von Neumann algebra case:** Results not quite as beautiful, and open questions start to appear...

**Theorem** Consider the inclusions $D \subset \mathcal{A} \subset M$, where $M$ is a $\sigma$-finite von Neumann algebra with faithful normal state $\varphi$, $\mathcal{A}$ is a weak* closed subalgebra of $M$, and $D$ is a von Neumann subalgebra containing the unit of $M$. Suppose that $\sigma_t^\varphi(\mathcal{A}) = \mathcal{A}$ and $\sigma_t^\varphi(D) = D$ for each $t \in \mathbb{R}$. Let $\Phi : \mathcal{A} \to \mathcal{D}$ be a weak* continuous completely bounded $D$-character which commutes with $(\sigma_t^\varphi)$. Then there exists a conditional expectation $\Psi$ from $M$ onto $D$ extending $\Phi$.

(Very technical, von Neumann algebraic, using Haagerup's reduction method. It should, but it does'nt quite ...)

Theorem Consider the inclusions $D \subset \mathcal{A} \subset M$, where $M$ is an arbitrary von Neumann algebra, $\mathcal{A}$ is a weak* closed subalgebra of $M$, and $D$ is a von Neumann subalgebra containing the unit of $M$. Suppose that $M$ is equipped with a fns weight $\varphi$ for which the restriction to $D$ is strictly semifinite, and that $\sigma_t^\varphi(\mathcal{A}) = \mathcal{A}$ and $\sigma_t^\varphi(D) = D$ for each $t \in \mathbb{R}$. Let $\Phi : \mathcal{A} \to \mathcal{D}$ be a weak* continuous completely bounded $D$-character which commutes with $(\sigma_t^\varphi)$. Then there exists a conditional expectation $\Psi$ from $M$ onto $D$ extending $\Phi$.

**Theorem** Consider the inclusions $D \subset \mathcal{A} \subset M$, where $M$ is an arbitrary von Neumann algebra, $\mathcal{A}$ is a weak* closed subalgebra of $M$, and $D$ is a von Neumann subalgebra containing the unit of $M$. Suppose that $M$ is equipped with a fns weight $\varphi$ for which the restriction to $D$ is strictly semifinite, and that $\sigma_t^\varphi(\mathcal{A}) = \mathcal{A}$ and $\sigma_t^\varphi(D) = D$ for each $t \in \mathbb{R}$. Let $\Phi : \mathcal{A} \to \mathcal{D}$ be a weak* continuous completely bounded $D$-character which commutes with $(\sigma_t^\varphi)$. Then there exists a conditional expectation $\Psi$ from $M$ onto $D$ extending $\Phi$.

- Von Neumann algebraic; however at this level of generality we do not know if $\Psi$ can be chosen to be normal.

And thank you for your attention!

D.P.B + L Labuschagne: Von Neumann algebra conditional expectations with applications to generalized representing measures for noncommutative function algebras, to appear Adv. in Mathematics.