An Indefinite Analog of Sarason's Generalized Interpolation Theorem

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Spaces $\mathcal{H}(C)$

If C analytic and bounded by one on $\mathbb{D}$, let $\mathcal{H}(C)$ be the Hilbert space with reproducing kernel

$$K_C(w,z) = \frac{1 - C(z)\overline{C(w)}}{1 - z\overline{w}}, \qquad w, z \in \mathbb{D}.$$

Equivalently, $\mathcal{H}(C)$ is the space of all $f(z)$ in H^2 such that

$$\|f(z)\|_C^2 = \sup \left[\|f(z) + C(z)g(z)\|^2 - \|g(z)\|^2 \right] < \infty.$$

Besides kernel functions, $\mathcal{H}(C)$ contains all difference quotients

$$\frac{f(z) - f(w)}{z - w} \quad \text{and} \quad \frac{C(z) - C(w)}{z - w}$$

whenever $f(z)$ is in the space and $w \in \mathbb{D}$.

Operators on $\mathcal{H}(C)$

Define contraction operators T and T^* on $\mathcal{H}(C)$ by

$$T^* : f(z) \to \frac{f(z) - f(0)}{z}$$

and

$$T : f(z) \to z\, f(z) - C(z) \left\langle f(z), \frac{C(z) - C(0)}{z} \right\rangle_C .$$

Special case: C an inner function

$$\mathcal{H}(C) = H^2 \ominus CH^2 \text{ (isometrically)}$$

T = compression of "multiplication by z" to $\mathcal{H}(C)$

Functions of T:

For any H^∞ function $f(z) = \sum_0^\infty f_j\, z^j$, the Sz.-Nagy and Foias functional calculus defines

$$f(T) = s\text{-}\lim_{r \uparrow 1} \sum_{j=0}^{\infty} f_j\, r^j T^j, \qquad \|f(T)\| \leq \|f\|_\infty$$

The same formula can be used for any completely nonunitary operator on a Hilbert space.

If S is multiplication by z on H^2, $f(S)$ is multiplication by $f(z)$.

When C is an inner function, then $T = P_C S|\mathcal{H}(C)$ and

$$f(T) = P_C f(S)\big|\mathcal{H}(C).$$

This is the form used by Sarason.

Theorem (Sarason 1967)

Let C be an inner function. If R is a bounded operator on $\mathcal{H}(C)$ that commutes with T, then there is an $f \in H^\infty$ such that

$$R = f(T)$$

Moreover f can be chosen such that $\|f\|_\infty = \|R\|$.

When R is a contraction, we can choose $f \in \mathbf{S}_0$:

Schur class $\mathbf{S}_0$: analytic functions f on $\mathbb{D}$ such that $|f(z)| \leq 1$.

Generalized Schur class $\mathbf{S}_\kappa$: quotients f/B, where f is a Schur function, B is a Blaschke product of degree κ, and f and B have no common zeros.

Such pairs f, B play a prominent role in this talk.

Outline of talk

Replace "R a contraction" by "$1 - RR^*$ has κ negative squares", or $\mathrm{sq}_-(1 - RR^*) = \kappa$.

Meaning: the negative spectrum of $1 - RR^*$ consists of eigenvalues of total multiplicity κ.

Nudel$'$man's problem

Such an R satisfies $B(T)R = f(T)$, where B is a Blaschke product of degree κ and f is a Schur function.

Preliminaries on spaces $\mathcal{H}(C)$

Difference-quotient identity and root subspaces for T and T^*.

Applications to $B(T)R = f(T)$

Goal: R is determined on a subspace of codimension at most κ and given by an explicit formula depending on f, B.

Nudel$'$man's problem

Abstract problem

Let $\mathcal{V}$ be a complex vector space, $A\colon \mathcal{V} \to \mathcal{V}$ a linear mapping. Define $A'\colon \mathcal{V}' \to \mathcal{V}'$ by

$$(Ax, x') = (x, A'x'), \qquad x \in \mathcal{V},\ x' \in \mathcal{V}'.$$

Choose a subspace $\mathcal{D} \subseteq \mathcal{V}'$ which is invariant under A'.

Problem

Let $b, c \in \mathcal{V}$ be given vectors, κ a nonnegative integer. Find a Schur function f and Blaschke product B of degree κ such that

$$B(A)b = f(A)c$$

in the sense that

$$\sum_{j=0}^{\infty} B_j\,(A^j b, x') = \sum_{j=0}^{\infty} f_j\,(A^j c, x'), \qquad x' \in \mathcal{D}.$$

Assume $\mathcal{D}$ **admissible** in the sense that for all $x' \in \mathcal{D}$,

$$\sum_{j=0}^{\infty} |(A^j b, x')|^2 \leq M \sum_{j=0}^{\infty} |(A^j c, x')|^2 < \infty.$$

Theorem (ADR 2020)

The problem admits a solution if the kernel on $\mathcal{D} \times \mathcal{D}$ defined by

$$\mathbf{K}(x', y') = \sum_{j=0}^{\infty} \left[(A^j c, x')\overline{(A^j c, y')} - (A^j b, x')\overline{(A^j b, y')} \right]$$

has κ negative squares.

Conversely, if a solution exists, $\mathbf{K}(x', y')$ has at most κ negative squares.

A kernel $K(w,z) = \overline{K(z,w)}$ on $\Omega \times \Omega$ has κ negative squares if among all matrices

$$\left(K(w_j, w_i)\right)_{i,j=1}^{n}, \quad w_1, \ldots, w_n \in \Omega,\ n \geq 1,$$

the maximum number of negative eigenvalues counting multiplicity is κ.

Proof and history: Alpay, Dijksma, Rovnyak 2020

Arocena, Azizov, Dijksma, Marcantognini 1997

Ball and Helton 1983

The case $\kappa = 0$ has a simpler statement and proof:

Rosenblum and Rovnyak 1985

Pick-Nevanlinna interpolation:

Let $z_1, \dots, z_n$ be distinct points in $\mathbb{D}$, $w_1, \dots, w_n \in \mathbb{C}$. Set

$$P = \left(\frac{1 - w_j \bar{w}_i}{1 - z_j \bar{z}_i} \right)_{i,j=1}^{n}.$$

If $\mathrm{sq}\_\ P = \kappa$, there is a Schur function f and a Blaschke product B of degree κ such that

$$B(z_j) w_j = f(z_j), \qquad j = 1, \dots, n.$$

Conversely, if such f and B exist, then $\mathrm{sq}\_\ P \leq \kappa$.

Choose $A = \mathrm{diag}\,\{z_1, \dots, z_n\}$ on $\mathcal{V} = \mathbb{C}^n$,

$$b = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Take $\mathcal{D} = \mathbb{C}^n$ with the pairing $(x, y) = x_1 y_1 + \cdots + x_n y_n$.

Theorem A (ADR 2020)

Assume C inner. Let R be an operator on $\mathcal{H}(C)$ that commutes with T and satisfies

$$\mathrm{sq}_-(1 - RR^*) = \kappa.$$

Then there exist a Schur function f and a Blaschke product B of degree κ such that

$$B(T)R = f(T).$$

Conversely, if such f and B exist, then $1 - RR^*$ has at most κ negative squares.

Special case: $\kappa = 0$.

For any $R \neq 0$ that commutes with T, we can apply Theorem A to $R/\|R\|$ and recover Sarason's theorem.

Sketch of proof:

Choose $\mathcal{V} = \mathcal{H}(C)$,

$$T \colon \mathcal{H}(C) \to \mathcal{H}(C)$$

$$c = K_C(0, z) \quad \text{and} \quad b = R\, K_C(0, z)$$

$$\mathcal{D} = \text{all continuous linear functionals on } \mathcal{H}(C)$$

Admissibility requires

$$\sum_{j=0}^{\infty} |(A^j b, x')|^2 \leq M \sum_{j=0}^{\infty} |(A^j c, x')|^2 < \infty.$$

Compute:

$$\sum_{j=0}^{\infty} \left| \langle T^j K_C(0,\cdot), k \rangle_C \right|^2 = \|k\|_{H^2}^2 = \|k\|_C^2,$$

$$\sum_{j=0}^{\infty} \left| \langle T^j R K_C(0,\cdot), k \rangle_C \right|^2 = \|R^* k\|_{H^2}^2 = \|R^* k\|_C^2.$$

The admissibility condition is met:

$$\|R^* k\|_C^2 \leq M \|k\|_C^2 < \infty.$$

By a similar calculation,

$$\begin{aligned}\mathbf{K}(x', y') &= \sum_{j=0}^{\infty} \left[(A^j c, x')\overline{(A^j c, y')} - (A^j b, x')\overline{(A^j b, y')} \right] \\ &= \langle (1 - RR^*)k, h \rangle_{H^2} \\ &= \langle (1 - RR^*)k, h \rangle_{C}\end{aligned}$$

has κ negative squares.

Hence there exist a Schur function f and Blaschke product B of degree κ such that

$$B(T)b = f(T)c, \qquad b = Rc.$$

Therefore $B(T)R$ and $f(T)$ agree on $K_C(0, z)$.

Since $K_C(0, z)$ is cyclic for T, $B(T)R = f(T)$.

Theorem A follows.

What remains to do?

By Theorem A, in Sarason's theorem, if $\mathrm{sq}_{-}(1 - RR^*) = \kappa$ then

$$B(T)R = f(T)$$

for some such f and B. What does this tell us about R?

Problem

Let R be a bounded operator on a space $\mathcal{H}(C)$, C not necessarily inner, that commutes with T and satisfies

$$B(T)R = f(T),$$

where f is a Schur function and B is a Blaschke product of degree κ. What is the form of R?

This is the topic of the rest of the talk.

Preliminaries on spaces $\mathcal{H}(C)$

The difference-quotient identity

Recall that $\mathcal{H}(C)$ is the Hilbert space with reproducing kernel

$$K_C(w,z) = \frac{1 - C(z)\overline{C(w)}}{1 - z\overline{w}}, \qquad w, z \in \mathbb{D}.$$

The difference-quotient inequality holds in every space $\mathcal{H}(C)$:

$$\left\| \frac{h(z) - h(0)}{z} \right\|_C^2 \leq \|h(z)\|_C^2 - |h(0)|^2.$$

In a large class of spaces, the inequality is always an equality.

Theorem

The difference-quotient identity

$$\left\| \frac{h(z) - h(0)}{z} \right\|_C^2 = \|h(z)\|_C^2 - |h(0)|^2$$

holds for every $h(z)$ in the space if and only if $C \notin \mathcal{H}(C)$.

Sarason showed that $C \notin \mathcal{H}(C)$ if and only if C is an extreme point of the unit ball in H^∞.

Other equivalent conditions and viewpoints:
Ball and Bolotnikov 2015
Operator Theory, Springer, Editor Daniel Alpay

Henceforth we shall assume that $C \notin \mathcal{H}(C)$, that is, we assume the extreme point case.

The difference-quotient identity facilitates calculations involving

$$T\colon\ h(z) \to zh(z) - C(z)\left\langle h(z), \frac{C(z)-C(0)}{z} \right\rangle_C$$

and inner products of difference quotients

$$\left\langle \frac{h(z)-h(w)}{z-w}, \frac{C(z)-C(0)}{z} \right\rangle_C \text{ and } \left\langle \frac{C(z)-C(w)}{z-w}, \frac{C(z)-C(0)}{z} \right\rangle_C$$

Two useful formulas:

SSPS, Problem 88. If $C \notin \mathcal{H}(C)$, then

$$\overline{C(\beta)}h(\alpha) = \bar\beta \left\langle h(z), \frac{C(z)-C(\beta)}{z-\beta} \right\rangle_C - (1-\alpha\bar\beta)\left\langle \frac{h(z)-h(\alpha)}{z-\alpha}, \frac{C(z)-C(\beta)}{z-\beta} \right\rangle_C$$

SSPS, Problem 89. If $C \notin \mathcal{H}(C)$, then

$$\left\langle \frac{C(z)-C(\alpha)}{z-\alpha}, \frac{C(z)-C(\beta)}{z-\beta} \right\rangle_C = \frac{1-\overline{C(\beta)}C(\alpha)}{1-\alpha\bar\beta}$$

Different method:

Alpay, Dijksma, Rovnyak, de Snoo 1997 (Th. 3.2.4, 3.2.5)

Special cases, very useful for calculations involving T:

If $C \notin \mathcal{H}(C)$, then

$$\left\langle \frac{h(z) - h(w)}{z - w}, \frac{C(z) - C(0)}{z} \right\rangle_C = -\overline{C(0)}h(w)$$

$$\left\langle \frac{C(z) - C(w)}{z - w}, \frac{C(z) - C(0)}{z} \right\rangle_C = 1 - \overline{C(0)}C(w)$$

Example:

Assume $C \notin \mathcal{H}(C)$. If $C(w) \neq 0$, then

$$(T - w)^{-1} \colon h(z) \to \frac{h(z) - h(w)C(z)/C(w)}{z - w}.$$

Cf. Sarason 1994, p. 42.

Proof. Set

$$k(z) = \frac{h(z) - h(w)C(z)/C(w)}{z - w}$$

$$= \frac{h(z) - h(w)}{z - w} - \frac{h(w)}{C(w)} \frac{C(z) - C(w)}{z - w}.$$

Then

$$(T - w)k(z) = (z - w)k(z) - C(z)\left\langle k(z), \frac{C(z) - C(0)}{z} \right\rangle_C$$

$$= h(z).$$

Terminology

If α is an eigenvalue of A, its geometric multiplicity is

$$\dim \ker (A - \alpha).$$

Any $f \neq 0$ such that $(A - \alpha)^n f = 0$ is a root vector.

If the subspace of all root vectors plus zero has finite dimension, this dimension is the algebraic multiplicity of A.

[Gohberg and Kreĭn, *Introduction to the Theory of . . .*]

Eigenvalues and eigenfunctions

Sarason 1994
Fricain and Mashreghi 2016
Garcia, Mashreghi, and Ross 2016

Root functions

Nikol′skiĭ, *Treatise on the Shift Operator*

Assume given $\mathcal{H}(C)$, $C \notin \mathcal{H}(C)$. The spectrum of T in $\mathbb{D}$ consists of isolated eigenvalues at the zeros of C in $\mathbb{D}$.

If $C(\alpha) = 0$, then

(1) α is an eigenvalue of T, and $\ker(T - \alpha) = \left[\dfrac{C(z)}{z - \alpha}\right]$

(2) $\bar{\alpha}$ is an eigenvalue of T^*, and $\ker(T^* - \bar{\alpha}) = \left[\dfrac{1}{1 - \bar{\alpha} z}\right]$.

Proposition 1

Let α be a zero of C of order n.

(1) *Let* $Q_j(z) = \dfrac{C(z)}{(z-\alpha)^j}$, $j = 1, \dots, n$. *Then*

$$(T - \alpha)Q_1 = 0,$$

$$(T - \alpha)Q_j = Q_{j-1}, \quad j = 2, \dots, n.$$

(2) *The subspaces* $\mathcal{R}_k = \ker\,(T-\alpha)^k$ *are given by*

$$\mathcal{R}_k = \begin{cases} [Q_1, \dots, Q_k], & k = 1, \dots, n, \\ \mathcal{R}_n, & k > n. \end{cases}$$

(3) *The geometric multiplicity of* α *is* 1, *the algebraic multiplicity is* n. *The root subspace is* $\mathcal{R}_n$.

Corollary

If $f \in H^\infty$ and $f(T)Q_k = 0$ for some $k = 1, \dots, n$, then f has a zero at α of order at least k.

Proof. Define $g \in H^\infty$ by

$$f(z) = f(\alpha) + f'(\alpha)(z-\alpha) + \cdots + \frac{f^{(k-1)}(\alpha)}{(k-1)!}(z-\alpha)^{k-1} + (z-\alpha)^k g(z)$$

Apply $f(T)$ to Q_k. Since $(T-\alpha)^k Q_k = 0$,

$$0 = f(\alpha)Q_k + f'(\alpha)Q_{k-1} + \cdots + \frac{f^{(k-1)}(\alpha)}{(k-1)!}Q_1.$$

Since $Q_1, \dots, Q_k$ are linearly independent,

$$f(\alpha) = f'(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0.$$

Proposition 2

Let α be a zero of C of order n.

(1) *Let* $P_j(z) = \dfrac{z^{j-1}}{(1-\bar{\alpha}z)^j}$, $j = 1, \ldots, n$. *Then*

$$(T^* - \bar{\alpha})P_1 = 0$$

$$(T^* - \bar{\alpha})P_j = P_{j-1}, \quad j = 2, \ldots, n.$$

(2) *The subspaces* $\widetilde{\mathcal{R}}_k = \ker\,(T^* - \bar{\alpha})^k$ *are given by*

$$\widetilde{\mathcal{R}}_k = \begin{cases} [P_1, \ldots, P_k], & k = 1, \ldots, n, \\ \widetilde{\mathcal{R}}_n, & k > n. \end{cases}$$

(3) *The geometric multiplicity of* $\bar{\alpha}$ *is* 1, *the algebraic multiplicity is* n. *The root subspace is* $\widetilde{\mathcal{R}}_n$.

Proposition 3

Let α be a zero of C of order n. For each $j = 1, \dots, n$,

$$P_j(z) = R(\alpha)^{*\,j-1} K_C(\alpha, z),$$

where

$$R(\alpha)\colon h(z) \to \frac{h(z) - h(\alpha)}{z - \alpha}.$$

Corollary

For each $k = 1, \dots, n$,

$$\left[\ker\left(T^* - \bar{\alpha}\right)^k\right]^\perp = \left\{h\colon h(\alpha) = h'(\alpha) = \cdots = h^{k-1}(\alpha) = 0.\right\}.$$

This is because

$$\left\langle h(z), R(\alpha)^{*\,j-1} K_C(\alpha, z)\right\rangle_C = \frac{h^{j-1}(\alpha)}{(j-1)\,!}$$

Proposition 4

If $\alpha_1, \ldots, \alpha_r$ are distinct zeros of C, then

$$\ker\left[(T^* - \bar{\alpha}_1)^{m_1} \cdots (T^* - \bar{\alpha}_r)^{m_r}\right] = \sum_{j=1}^{r} \ker (T^* - \bar{\alpha}_j)^{m_j}.$$

for all positive integers $m_1, \ldots, m_r$.

The proof is by induction on the number of factors.

Cases:

1. Add another zero α_{n+1} of C.
2. Increase one m_j to $m_j + 1$ when $m_j + 1$ is not beyond the order of α_j as a zero of C.
3. Increase one m_j beyond this order.

In each case, we make a judicious partial fraction decomposition.

Applications to $B(T)R = f(T)$

Problem

Let R be a bounded operator on $\mathcal{H}(C)$, $C \notin \mathcal{H}(C)$, that commutes with T and satisfies

$$B(T)R = f(T),$$

where f is a Schur function, B a Blaschke product of degree κ.

What is the form of R? Is R uniquely determined?

Easy case:

$$\{\text{Zeros of B}\} \cap \{\text{Zeros of C}\} = \emptyset$$

Then $B(T)$ is a product of factors $(1 - \bar{\beta}T)^{-1}(T - \beta)$ such that $C(\beta) \neq 0$. Then $T - \beta$ is invertible.

Hence $B(T)$ is invertible, and so

$$R = B(T)^{-1} f(T).$$

In the general case, set

$$\{\textit{Zeros of } B\} \cap \{\textit{Zeros of } C\} = \{\alpha_1, \dots, \alpha_r\}$$

$$m_1, \dots, m_r = \text{orders of } \alpha_1, \dots, \alpha_r \text{ as zeros of } B$$

$$n_1, \dots, n_r = \text{orders of } \alpha_1, \dots, \alpha_r \text{ as zeros of } C$$

Factor

$$B(z) = B_1(z) B_0(z) B_2(z)$$

where

$$B_1(z) = \prod_{C(\beta)\neq 0} \frac{z-\beta}{1-\bar{\beta}z}$$

$$B_0(z) = \prod_{j=1}^{r} \left(\frac{z-\alpha_j}{1-\bar{\alpha}_j z} \right)^{k_j}, \qquad k_j = \min(m_j, n_j),$$

$$B_2(z) = \prod_{m_j > n_j} \left(\frac{z-\alpha_j}{1-\bar{\alpha}_j z} \right)^{m_j - n_j}$$

Theorem B

Let R be a bounded operator on $\mathcal{H}(C)$, $C \notin \mathcal{H}(C)$, that commutes with T and satisfies

$$B(T)R = f(T),$$

where f is a Schur function and B is a Blaschke product of degree κ. Factor $B = B_1 B_0 B_2$ as just described.

Let $\mathcal{K}$ be the subspace of all $h(z)$ in $\mathcal{H}(C)$ such that

$$h(\alpha_j) = h'(\alpha_j) = \cdots = h^{(k_j - 1)}(\alpha_j) = 0, \qquad j = 1, \dots, r.$$

Then $\operatorname{codim} \mathcal{K} \le \kappa$, and $\mathcal{K}$ is invariant under T and R.

The restriction $R_{\mathcal{K}} = R|\mathcal{K}$ is a function of $T_{\mathcal{K}} = T|\mathcal{K}$ given by

$$R_{\mathcal{K}} = B_1(T_{\mathcal{K}})^{-1} g(T_{\mathcal{K}}) B_2(T_{\mathcal{K}})^{-1},$$

where $g(z) = f(z)/B_0(z)$ is a Schur function.

Idea of the proof

We show that $g = f/B_0 \in \mathbf{S}_0$.

Since $B(T)R = f(T)$ and $B = B_1 B_0 B_2$,

$$B_1(T)B_0(T)B_2(T)R = f(T)$$

All factors on the left commute, and $B_1(T)$ is invertible because the zeros of B_1 are not zeros of C.

Therefore

$$RB_2(T)B_0(T) = B_1(T)^{-1}f(T).$$

Here

$$B_0(T) = \prod_{j=1}^{r} \left[(1 - \bar{\alpha}_j T)^{-1}(T - \alpha_j)\right]^{k_j}$$

By Proposition 1, $\ker(T - \alpha_j)^{k_j} = [Q_1, \dots, Q_{k_j}]$.

Hence $B_0(T)Q_{k_j} = 0$ and so $f(T)Q_{k_j} = 0$.

Corollary: f has a zero at α_j of order at least k_j.

By standard function theory,

$$f(z) \Big/ \left(\frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \right)^{k_j}$$

is a Schur function.

Since $B_0(z)$ is the product of all such factors,

$$g(z) = f(z)/B_0(z)$$

is a Schur function.

It follows that

$$RB_2(T)B_0(T) = B_1(T)^{-1}g(T)B_0(T).$$

This says that

$$RB_2(T)\Big|\operatorname{ran} B_0(T) = B_1(T)^{-1}g(T)\Big|\operatorname{ran} B_0(T)$$

Now

$$\begin{aligned}\overline{\operatorname{ran} B_0(T)} &= \Big[\ker B_0(T)^*\Big]^{\perp} \\ &= \Big[\ker\big[(T^* - \bar{\alpha}_1)^{k_1}\cdots(T^* - \bar{\alpha}_r)^{k_r}\big]\Big]^{\perp} \\ &= \Big[\sum_{j=1}^{r}\ker(T^* - \bar{\alpha}_j)^{k_j}\Big]^{\perp}\end{aligned}$$

by Proposition 4.

By Proposition 3,

$$\Big[\ker\,(T^* - \bar{\alpha}_j)^{k_j}\Big]^{\perp} = \Big\{h\colon h(\alpha_j) = h'(\alpha_j) = \cdots = h^{k_j-1}(\alpha_j) = 0\Big\}$$

Therefore $\overline{\operatorname{ran} B_0(T)} = \mathcal{K}$.

Clearly $T\mathcal{K} \subseteq \mathcal{K}$, and

$$\operatorname{codim}\mathcal{K} = k_1 + \cdots + k_r \leq \kappa.$$

We have shown

$$RB_2(T)|\mathcal{K} = B_1(T)^{-1}g(T)|\mathcal{K}$$

Short step to

$$R_{\mathcal{K}} B_2(T_{\mathcal{K}}) = B_1(T_{\mathcal{K}})^{-1} g(T_{\mathcal{K}}).$$

It remains to move $B_2(T_{\mathcal{K}})$ to the right side.

Claim: $B_2(T_{\mathcal{K}})$ is an invertible element of $\mathcal{L}(\mathcal{K})$

Omit

Granting this,

$$R_{\mathcal{K}} = B_1(T_{\mathcal{K}})^{-1} g(T_{\mathcal{K}}) B_2(T_{\mathcal{K}})^{-1}.$$

Theorem B follows.

Theorem C

Let R be an operator on $\mathcal{H}(C)$, C inner, that commutes with T and satisfies

$$\operatorname{sq}_{-}(1 - RR^*) = \kappa.$$

Then there is a Schur function f and a Blaschke product B of degree κ such that $B(T)R = f(T)$. Factor $B = B_1 B_0 B_2$ as in Theorem B.

Let $\mathcal{K}$ be the subspace of all $h(z)$ in $\mathcal{H}(C)$ such that

$$h(\alpha_j) = h'(\alpha_j) = \cdots = h^{(k_j-1)}(\alpha_j) = 0, \qquad j = 1, \dots, r.$$

Then $\operatorname{codim} \mathcal{K} \le \kappa$, and $\mathcal{K}$ is invariant under T and R.

The restriction $R_{\mathcal{K}} = R|\mathcal{K}$ is a function of $T_{\mathcal{K}} = T|\mathcal{K}$ given by

$$R_{\mathcal{K}} = B_1(T_{\mathcal{K}})^{-1} g(T_{\mathcal{K}}) B_2(T_{\mathcal{K}})^{-1},$$

where $g(z) = f(z)/B_0(z)$ is a Schur function.

Remark

In general, $\mathcal{K} \neq \mathcal{H}(C)$ and R cannot be known completely.

This is expected behavior in indefinite interpolation.

Unexpected behavior:

If C is a singular inner function, then automatically

$$B(T) = \prod_{C(\beta)\neq 0} (1 - \bar{\beta}T)^{-1}(T - \beta)$$

is invertible. Then $\mathcal{K} = \mathcal{H}(C)$ and $R = B(T)^{-1}f(T)$.

Thank you!