Isometric Extensions

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Typical interpolation problems

Given $V \subset \Omega$, and Banach function spaces X on V, Y on Ω.
For $f \in X$, find $\phi \in Y$ such that $\phi|_V = f$.

- Can this always be solved?
- What is best constant so that can always have $\|\phi\|_Y \leq C\|f\|_X$?
- For fixed f, what is smallest norm of extension ϕ?

Common phenomenon: Best ϕ often has extra regularity.

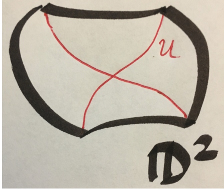
Variant interpolation problem

Suppose $C = 1$. What does this say about X and Y (or V and Ω)?

Sometimes not much - eg. Tietze extension theorem. But if you look at holomorphic functions, picture changes.

Prehistory - late 20th century

- Solve a Pick problem on bidisk $\mathbb{D}^2$ (find function with smallest H^∞ norm satisfying finitely many interpolation conditions)
- Either solution is unique, or there exists one dimensional variety $\mathcal{U}$ on which all solutions coincide
- All solutions satisfy $\|\phi\|_{\mathcal{U}} = \|\phi\|_{\mathbb{D}^2}$.



Does this say $\mathcal{U}$ is special, or $\phi|_{\mathcal{U}}$ is special?
Does every function in $H^\infty(\mathcal{U})$ extend to a function in $H^\infty(\mathbb{D}^2)$ of same norm?

Ω pseudo-convex domain in $\mathbb{C}^d$, $O(\Omega) :=$ holomorphic functions on Ω
V analytic subset of Ω
(locally defined as common zero set of functions in $O(\Omega)$)

Def: $f : V \to \mathbb{C}$ is holomorphic if $\forall \lambda \in V$, $\exists \varepsilon > 0$ and $h \in O(\mathbb{B}(\lambda, \varepsilon))$ with $h|_{V \cap \mathbb{B}(\lambda,\varepsilon)} = f|_{V \cap \mathbb{B}(\lambda,\varepsilon)}$

Q1 : Given $f \in O(V)$, is there a single h holomorphic on nbhd of V extending f? If so, can h be chosen in $O(\Omega)$?

A1: Yes always - H. Cartan, 1950

Isometric extension property

Q2: Which $V \subseteq \mathbb{D}^2$ have isometric extension property (IEP):
$\forall\, f \in H^\infty(V)\ \exists\, \phi \in H^\infty(\mathbb{D}^2)$, norm-preserving extension

Example

$\Omega = \mathbb{D}^2,\ V = \{z \in \mathbb{D}^2 : z_1 z_2 = 0\}$
$f(z_1, 0) = z_1,\ f(0, z_2) = z_2$.
$\|D\phi(0)\| = \|(1,1)\| = \sqrt{2}$. Contradicts Schwarz's Lemma.
Singularities bad

In general, answer to Q2 not known. But for nice sets (eg algebraic sets)

Thm. [Agler-M 2003]

If $V \subseteq \mathbb{D}^2$ is polynomially convex, then it has IEP iff it is a retract.

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Thm. [Agler-M 2003]

If $V \subseteq \mathbb{D}^2$ is polynomially convex, then it has IEP iff it is a retract.

Retract

Def: V is a retract of Ω if $\exists\, r : \Omega \to V$, holomorphic, $r|_V = \mathrm{id}$.
If V is retract, $\phi := f \circ r$ gives norm-preserving extension.

Thm. [Heath-Suffridge 1981]

All retracts of $\mathbb{D}^d$ are graphs
$\{(z, \Psi(z)) : z \in \mathbb{D}^m, \Psi : \mathbb{D}^m \to \mathbb{D}^{d-m} \text{ holomorphic}\}$

Question R

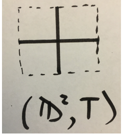
If V is a polynomially convex analytic subset of Ω with IEP, is V a retract?

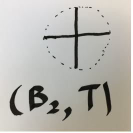
Ans R: Yes if

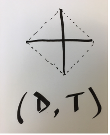
- $\Omega = \mathbb{D}^2$
- $\Omega = \mathbb{B}_d$ (Kosinski-M 19)
- Ω is strictly convex and 2-dimensional (Kosinski-M 19)

No if

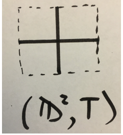
- Ω is symmetrized bidisk (not convex) (Agler-Lykova-Young 17)
- $D = \{|z_1| + |z_2| < 1\}$ (convex, not strictly convex)

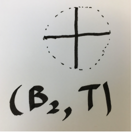


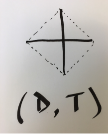




$T := \mathbb{D} \times \{0\} \cup \{0\} \times \mathbb{D}$







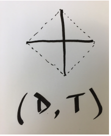
Schwarz lemma for balanced set Ω

Suppose $\phi : \Omega \to \mathbb{D}$ and $\phi(0) = 0$.
Then $D\phi(0) : \Omega \to \mathbb{D}$

Ω is balanced if $\lambda \in \Omega \Rightarrow z\lambda \in \Omega \ \forall \ z \in \overline{\mathbb{D}}$

Schwarz Lemma rules out first two

$f(z_1, 0) = z_1, \ f(0, z_2) = z_2 \qquad \Rightarrow D\phi(0) = Df(0) = (1, 1)$
$(1, 1)$ does not map $\mathbb{D}^2$ or B_2 to $\mathbb{D}$.
It does map D to $\mathbb{D}$



Schwarz Lemma goes from enemy to friend

Thm: (D, T) has IEP

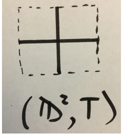
Let $g \in H^\infty(T)$, $\|g\| \leq 1$ and suppose $g(0) = 0$.

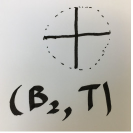
$$E(g) = \phi(z_1, z_2) := g(z_1, 0) + g(0, z_2)$$

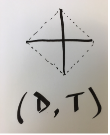
Win by Schwarz!

$$|g(z_1, 0) + g(0, z_2)| \leq |g(z_1, 0)| + |g(0, z_2)| \leq |z_1| + |z_2| < 1$$

If $f(0) = a$, use $m_a \circ E(m_a \circ f)$, where $m_a(z) = \frac{a-z}{1-\bar{a}z}$.







- What is this terrifying anomaly?

Shift Perspective

Prob A: Given Ω, find all V s.t. (Ω, V) has IEP
Prob B: Given V find all Ω s.t. (Ω, V) has IEP

What conditions must V satisfy for $\{\Omega : (\Omega, V) \text{ has IEP}\}$ non-empty?

Thm 1 [Agler-Kosinski-M]

Let V be an analytic subset of some domain of holomorphy. Then $\exists$ domain of holomorphy Ω s.t. (Ω, V) has IEP.

If want Ω to be connected, then V must be too (maximum principle).

Most sets (eg T, the two crossed disks) are not retracts of anything.

Absent some form of convexity, retracts seem to have little to do with Isometric Extension Property

Analyze $T := \mathbb{D} \times \{0\} \cup \{0\} \times \mathbb{D}$

Thm 2 [Agler-Kosinski-M]

Let Ω be balanced pseudoconvex domain in $\mathbb{C}^2$ with $T \subset \Omega$.
Then (Ω, T) has IEP iff $\Omega \subseteq D = \{|z_1| + |z_2| < 1\}$.

Dropping balanced it gets more complicated.

Thm 3 [Agler-Kosinski-M]

Let Ω be pseudoconvex domain in $\mathbb{C}^2$ with $T \subset \Omega$.
Then (Ω, T) has IEP iff T is relatively closed in Ω and $\exists$ pseudoconvex set G in $\mathbb{C}^2$ and a function $\tau \mapsto C_\tau$ from $\mathbb{T}^2$ into $\mathrm{Hol}(G)$ so that

$$\Omega = \cap_{\tau \in \mathbb{T}^2} \{\lambda \in G : |\tau \cdot \lambda + \lambda_1 \lambda_2 C_\tau(\lambda)| < 1\}.$$

Says just need to be able to extend each $\tau \cdot \lambda = \tau_1 \lambda_1 + \tau_2 \lambda_2$

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Example

Choose $G = \mathbb{D}^2$ and $C_\tau(\lambda) = \tau \cdot \lambda$.

$$\Omega := \{z \in \mathbb{D}^2 : (|z_1| + |z_2|)|1 + z_1 z_2| < 1\}$$

(Ω, T) has IEP, $\Omega \not\subset D$ and $D \not\subset \Omega$.

No maximal domain (without balanced)

Question (Rudin, 1969)

If (Ω, V) has bounded extension property (every $f \in H^\infty(V)$ extends to $H^\infty(\Omega)$, but with perhaps larger norm), is there a bounded linear operator?

Don't know, but no for isometric

Thm 4 [Agler-Kosinski-M]

There is no isometric linear extension operator from $H^\infty(T)$ to $H^\infty(D)$.

Can do it linearly with smaller domain

Thm 5 [Agler-Kosinski-M]

There is a domain Ω containing T and an isometric linear extension operator from $H^\infty(T)$ to $H^\infty(\Omega)$.

Can, using operator theory, analyze some other sets

Example

$$\mathcal{V} = \{z \in \mathbb{D}^3 : z_3^2 = z_1 z_2\}$$

This is a branched cover of the bidisk inside the tridisk.
What is an isometric envelope?

Can, using operator theory, analyze some other sets

Example

$$\mathcal{V} = \{z \in \mathbb{D}^3 : z_3^2 = z_1 z_2\}$$

$$\mathcal{G} = \{|z_1 z_2 - z_3|^2 < (1 - |z_3|^2) + \sqrt{1 - |z_1|^2}\sqrt{1 - |z_2|^2}\}$$

Thm 6 [Agler-Kosinski-M]

$\mathcal{G}$ is convex, and $(\mathcal{G}, \mathcal{V})$ has IEP.

If Ω is balanced, $(\Omega, \mathcal{V})$ has IEP iff $\Omega \subseteq \mathcal{G}$.

Challenge

Prove Theorem 6 without using operator theory!

Can, using operator theory, analyze some other sets

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Questions

Question 1 - Linearity

Let V be an analytic set. Does there always exist a domain Ω s.t. (Ω, V) has the IEP with a linear extension operator?

Question 2 - Complete isometric extensions

Suppose (Ω, V) has IEP. Does it have complete isometric extension property?

Question 3 - Back to retracts

Suppose $(\mathbb{D}^3, V)$ has IEP and V is relatively polynomially convex. Is V a retract?

Question 4 - Rule this out

$$V = \{z \in \mathbb{D}^3 : z_1 + z_2 + z_3 = z_1 z_2 + z_1 z_3 + z_2 z_3\}$$

Does $(\mathbb{D}^3, V)$ have IEP?

Thank You!

What does convexity have to do with retracts?

Thm (KM): If Ω is strictly convex and 2-dimensional, then (Ω, V) has IEP iff V is retract.

A geodesic map is a holomorphic $k : \mathbb{D} \to \Omega$ with a left inverse $c : \Omega \to \mathbb{D}$. (Also called Kobayashi extremal)

A set $G \subseteq \Omega$ is geodesically complete if, whenever k is a geodesic map and $k(\lambda_1), k(\lambda_2) \in G$, then $k(\mathbb{D}) \subseteq G$. (Or $k(\lambda_1)$ and tangent vector)

Step 1: If Ω is strictly convex and V has IEP, then V is geodesically complete.

Step 2: $k(\mathbb{D})$ is a retract (since $r = k \circ c$ is retraction)

If V is one dimensional, it is one geodesic. If Ω is 2-dimensional, 0 and 2 dimensional cases are trivial.