Algebraic twists of automorphic L-functions

Ph. Michel, EPF Lausanne
a series of joint works with
E. Fouvry E. Kowalski, Y. Lin and W.Sawin

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Let $(\lambda(n))_{n\geq 1}$ be the coefficients of an automorphic L-function of degree d

$$L(\pi, s) = \sum_{n\geq 1} \frac{\lambda(n)}{n^s} = \prod_p L_p(\pi, s),\ \Re s > 1$$

$$L_p(\pi, s) = \prod_{i\leq d} (1 - \frac{\alpha_{i,p}}{p^s})^{-1}.$$

To simplify we assume that the Ramanujan-Petersson bound holds

$$\lambda(n) = n^{o(1)}$$

(and often this known on average)

- The Godement-Jacquet L-function $L(\pi, s)$ of an automorphic representation $\pi \in \mathrm{Aut}(\mathsf{GL}_d)$,
- The Rankin-Selberg L-function, $L(\pi_1 \otimes \pi_2, s)$ of a pair $(\pi_1, \pi_2) \in \mathrm{Aut}(\mathsf{GL}_{d_1}) \times \mathrm{Aut}(\mathsf{GL}_{d_2})$.
- The L-function $L(\pi, \rho, s)$ attached to $\pi \in \mathrm{Aut}(\mathsf{GL}_d)$ and ρ a representation of $\mathsf{GL}_d(\mathbb{C})$: for instance $L(\pi, \mathrm{Ad}, s)$.
- It is expected (Functoriality Conjecture) that all are products of Godement-Jacquet L-functions

Let $\chi \pmod{q}$ be a Dirichlet character (of prime conductor) and $L(\pi, s)$ as above, the twisted L-function is

$$L(\pi.\chi, s) = \sum_{n \geq 1} \frac{\lambda(n)\chi(n)}{n^s} = \prod_p L_p(\pi.\chi, s), \ \Re s > 1$$

$$L_p(\pi.\chi, s) = \prod_{i \leq n} (1 - \frac{\alpha_{i,p}.\chi(p)}{p^s})^{-1}.$$

Assume the conductor of $L(\pi, s)$ is 1 (for simplicity) then

$$\Lambda(\pi.\chi, s) = \varepsilon.\varepsilon(\chi)^d \overline{\Lambda(\pi.\chi, 1 - \overline{s})}$$

where

$$\Lambda(\pi.\chi, s) = q^{ds/2} L_\infty(\pi.\chi, s) L(\pi.\chi, s)$$

is the complete L-function, $|\varepsilon| = 1$ and $\varepsilon(\chi)$ is the normalized Gauss sum.

Subconvexity Problem

Show that there is $\delta = \delta_d > 0$ such that for $\Re s = 1/2$,

$$L(\pi.\chi, s) \ll_{\pi,s} q^{d/4-\delta+o(1)},\ \delta > 0.$$

- GL_1: Burgess, Conrey-Iwaniec, Petrow-Young.
- GL_2: Duke-Friedlander-Iwaniec,..., Conrey-Iwaniec,..., Petrow-Young,..., Venkatesh,.., Nelson...
- $\mathrm{GL}_2 \times \mathrm{GL}_2$: M, Harcos-M,...,M-Venkatesh, ???
- $\mathrm{sym}(\mathrm{GL}_2)$: Blomer...
- GL_3: Munshi, Holowinsky-Nelson, Sharma...
- $\mathrm{GL}_2 \times \mathrm{sym}(\mathrm{GL}_2)$: Blomer...
- $\mathrm{GL}_2 \times \mathrm{GL}_3$: Sharma...

The problem is substantially equivalent to the bound

$$\sum_{n\sim q^{d/2}} \lambda(n)\chi(n) \ll_\pi q^{d/2-\delta+o(1)}.$$

– A fews years ago, F, K and M looked at the problem of establishing similar bounds with $\chi \pmod q$ replaced by more general q-periodic arithmetic functions $K : \mathbb{Z} \mapsto \mathbb{C}$ such that $\|K\|_\infty \ll 1$.
– The aim is to bound non trivially, sums of the shape

$$\sum_{n\sim X} \lambda(n)K(n) \ll_\pi X^{1-\delta},\ \delta > 0,$$

at least for some ranges of X wrt to q:

$$X_c = q^{d/2}$$

is the *convexity* range.
–The class of functions K considered are the so-called trace functions.

Trace functions include

- Additive characters $n \mapsto e_q(a.n) := e(\frac{an}{q})$,
- Dirichlet characters $n \mapsto \chi(n)$,
- composition of the above with rational fractions: for instance the "Kloosterman fraction",

$$\mathrm{Kl}_{-1}(n) := n \mapsto e_q(\frac{\overline{n}}{q}),\ (n, q) = 1$$

- Hyper-Kloosterman sums:

$$\mathrm{Kl}_k(n; q) = \frac{1}{q^{\frac{k-1}{2}}} \sum_{x_1 x \cdots x_k = n} e_q(x_1 + \cdots + x_k).$$

To a trace function K is attached a "conductor" $c(K)$ which we assume is bounded independently of q:

$$\Longrightarrow \|K\|_\infty \ll 1$$

λ in (large) arithmetic progressions

One possible motivation is the distribution of λ in large arithmetic progressions:

$$\sum_{\substack{n\sim N\\ n\equiv a \,(\mathrm{mod}\, q)}} \lambda(n) = \frac{1}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\chi}(a) \sum_{n\sim N} \lambda(n)\chi(n)$$

$$= \frac{1}{\varphi(q)} \sum_{\substack{n\sim N\\ (n,q)=1}} \lambda(n) + \frac{1}{\varphi(q)} \sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0}} \overline{\chi}(a) \sum_{n\sim N} \lambda(n)\chi(n)$$

and

$$\sum_{n\sim N} \lambda(n)\chi(n) \approx \frac{N}{q^{d/2}} \varepsilon(\chi)^d \sum_{n\ll q^d/N} \overline{\lambda(n)}\overline{\chi}(n)$$

$$\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \cdots \approx \frac{N}{q^{d/2}} \sum_{n \ll q^d/N} \overline{\lambda(n)} \Big(\frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \varepsilon(\chi)^d \overline{\chi}(an) \Big)$$

$$\frac{1}{\varphi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \varepsilon(\chi)^d \overline{\chi}(an) = \frac{1}{q^{1/2}} \mathrm{Kl}_d(an) + O(q^{-d/2-1})$$

One eventually obtains

$$\sum_{\substack{n \sim N \\ n \equiv a \pmod q}} \lambda(n) - \frac{1}{\varphi(q)} \sum_{\substack{n \sim N \\ (n,q)=1}} \lambda(n) \approx \frac{N}{q^{(d+1)/2}} \sum_{n \ll q^d/N} \overline{\lambda}(n) \mathrm{Kl}_d(an; q).$$

- The trivial bound (using Deligne's bound $|\mathrm{Kl}_d(an;q)| \leq d$) gives

$$\sum_{\substack{n\sim N\\ n\equiv a \pmod q}} \lambda(n) - \frac{1}{\varphi(q)} \sum_{\substack{n\sim N\\ (n,q)=1}} \lambda(n) \ll q^{\frac{d-1}{2}+o(1)}$$

which is better than $N^{1+o(1)}/q$ as long as $q \leq N^{\theta_d - \eta}$ where

$$\theta_d = \frac{2}{d+1}.$$

- In other terms, non-trivial bounds

$$\sum_{n\sim X} \lambda(n)\mathrm{Kl}_d(an;q) \ll_\pi X^{1-\delta},\ \delta > 0,$$

for the shorter range

$$X \gg X_d = q^{\frac{d}{2}-\frac{1}{2}}$$

will improve the level of distribution of λ in arithmetic progressions.

This problem is an analog of providing good estimates for sums with a sharp cut :

$$\sum_{n\leq X}\lambda(n)=\mathrm{res}_{s=1}\frac{L(\pi,s)X^s}{s}+\mathrm{Err}_\pi(X).$$

- Friedlander-Iwaniec have shown that in fair generality one can take
$$\mathrm{Err}_\pi(X)=O_\varepsilon(X^{\frac{d-1}{d+1}+\varepsilon}),\ \varepsilon>0.$$
- Going beyond this generic bound lead to the problem of bounding non-trivially sums of the shape

$$\sum_{n\sim X}\lambda(n)\exp(2\pi iT\varphi(n/X))$$

where $\varphi(x)=x^\beta$ and for suitable ranges of T wrt X

For instance, in the case of $\mathrm{GL}_2 \times \mathrm{GL}_2$ Rankin-Selberg L-functions ($d = 4$), one has

Theorem (B. Huang)

For f a modular form of level 1

$$\sum_{n \leq X} \lambda_f(n)^2 = \frac{L(\mathrm{sym}^2 f, 1)}{\zeta(2)} X + O_f(X^{3/5-1/561})$$

This improves the 80 years old $3/5$ exponent of Rankin and Selberg.

GL_1

The problem is essentially about bounding non-trivially

$$\sum_{n \sim X} K(n).$$

The convexity range here $X = q^{1/2}$ is also called the *Polya-Vinogradov* range and non-trivial bounds exists only for very limited amount of trace functions

1. $n \mapsto e_q(P(n))$, $P \in \mathbb{Z}[X]$ of degree ≥ 3: Weyl.
2. $n \mapsto \chi(n)$: Burgess.
3. $n \mapsto \chi(n) e_q(P(n))$: Burgess, Enflo,... ,Chang,... Heath-Brown-Pierce.
4. The Fourier transforms of the above: FKM-Raju-Rivat-Soundararajan.

$$n \mapsto \widehat{K}(n) = \frac{1}{q^{1/2}} \sum_{x \pmod q} K(x) e_q(nx) :$$

GL_2

For $(\lambda(n))_n$ the Fourier coefficients of a modular form, one has

$$X_c = q,\ \theta_2 = 2/3,\ X_d = q^{1/2}.$$

Theorem (FKM)

As long as K "is not" an additive character $n \mapsto e(\frac{an}{q})$, one has

$$\sum_{n \sim X} \lambda(n)K(n) \ll_{\pi, c(K)} (qX)^{o(1)} q^{1/2-1/8} X^{1/2}.$$

- For $X = X_c = q$ the bound saves $q^{1/8}$.
- The bound is non trivial as long as $X \geq q^{3/4+o(1)}$. So this does not improve θ_2.

The proof uses amplification "à la" Bykovskii/Conrey-Iwaniec:

$$q^{-1+o(1)}|\sum_{\ell\leq L} x_\ell \lambda_{f_0}(\ell)|^2.|\sum_{n\leq X}\lambda_{f_0}(n)K(n)|^2$$
$$\leq \sum_{f\in B_k(q)}|\sum_{\ell\leq L} x_\ell\lambda_f(\ell)|^2|\sum_{n\leq X}\lambda_f(n)K(n)|^2$$

followed by Kuznetzov formula and Poisson summation in the resulting two "n" variables.

Ultimately the bound rest on bounding the following correlation sums

$$\sum_{x\in\mathbb{F}_q}\widehat{K}(\gamma.x)\overline{\widehat{K}(x)}\ll_{c(K)}\delta_{\gamma\text{ bad}}\,q+q^{1/2}$$

where the

$$\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{PGL}_2(\mathbb{F}_q),\ \gamma.z=\frac{az+b}{cz+d}.$$

are constructed from the amplifier.
The goal is to show that the set of "bad" γ's is small.

- The bound above is a consequence of Deligne's Weil II and of a deep fact, due to Laumon, that unless K "is" an additive character, the Fourier transform $\widehat{K}$ is a trace function whose conductor $c(\widehat{K})$ is controlled by $c(K)$.

Theorem (Classification of group of automorphisms of sheaves)

The set of $\gamma \in \mathrm{PGL}_2(\mathbb{F}_q)$ such that the correlation sum is $\gg q$ is contained in $G_{\widehat{K}}(\mathbb{F}_q)$ the set of $\mathbb{F}_q$-points of an algebraic subgroup of PGL_2. Moreover $|G_{\widehat{K}}(\mathbb{F}_q)|$ is either "small" (bounded in terms of $c(K)$) or has a simple structure.

GL_3

We have

$$X_c = q^{3/2},\ \theta_3 = 1/2,\ X_d = q.$$

Theorem (KLMS)

As long as K "is not" an additive character $n \mapsto e(\frac{an}{q})$, one has

$$\sum_{n \sim X} \lambda(n) K(n) \ll_{\pi, c(K)} (qX)^{o(1)} q^{2/9} X^{5/6}.$$

- For $X = q^{3/2}$ one obtains $\ll_{\pi, c(K)} q^{3/2 - 1/36 + o(1)}$
- The bound is non trivial as long as $X \geq q^{4/3 + o(1)}$. So this does not improve θ_3.

GL_3: improving θ_3 in special cases

However if the GL_3-representation π is not cuspidal one can improve the performance using the fact that λ admits a factorisation

$$\lambda(n) = \lambda_1 \star \lambda_2(n) = \sum_{kl=n} \lambda_1(k)\lambda_2(l)$$

for $\lambda_i,\ i = 1, 2$ associated to $\pi_i \in \mathrm{Aut}(\mathrm{GL}_i)$.

- $\pi = 1 \boxplus 1 \boxplus 1$: $\lambda(n) = 1 \star 1 \star 1(n) = d_3(n)$, Friedlander-Iwaniec improved θ_3 by $1/230$, Heath-Brown by $1/80$ and FKM by $1/46$ (using the GL_2 case for $\pi_2 = 1 \boxplus 1$).
- $\pi = 1 \boxplus \pi_2$: for $\lambda(n) = 1 \star \lambda_f(n)$ KMS improved θ_3 by $1/102$. The key input is a bound for bilinear sums of Kloosterman sums in the PV range

$$\sum_{k,l \simeq q^{1/2}} \alpha_k \beta_l . \mathrm{Kl}_d(akl; q) \ll_d q^{1-1/64+o(1)}.$$

GL_3: improving θ_3 in special cases

However if the GL_3-representation is not cuspidal one can improve the performance using the fact that λ admits a factorisation

$$\lambda(n) = \lambda_1 \star \lambda_2(n) = \sum_{kl=n} \lambda_1(k)\lambda_2(l)$$

for $\lambda_i,\ i = 1, 2$ associated to $\pi_i \in \mathrm{Aut}(\mathrm{GL}_i)$.

- For $K = \mathrm{Kl}_k$, one has (KMS)

$$\sum_{n\sim X} \lambda_1 \star \lambda_2(n)\mathrm{Kl}_k(n) \ll X^{1-\delta}$$

  for $X \geq q^{3/4+\eta}$.

GL_3: the cuspidal case

After a first breakthrough made by X. Li, R. Munshi developed the δ-symbol method to obtain

Theorem (Munshi)

Let φ be a $SL_3(\mathbb{Z})$-invariant cusp form. For $K(n) = \chi(n)$,

$$\sum_{n \sim q^{3/2}} \lambda_\varphi(1,n)K(n) \ll_f q^{3/2-\delta+o(1)}, \ \delta = 1/308.$$

Holowinsky-Nelson simplified Munshi's approach

Theorem (Holowinsky-Nelson)

Let φ be a $SL_3(\mathbb{Z})$-invariant cusp form. For $K(n) = \chi(n)$,

$$\sum_{n \sim q^{3/2}} \lambda_\varphi(1,n)K(n) \ll_f q^{3/2-\delta+o(1)}, \ \delta = 1/36.$$

GL_3: the cuspidal case

Munshi's method can be adapted to handle more general trace functions K but at the cost of massive complications (also on the ℓ-adic side).
Fortunately the Holowinski-Nelson simplification also works and is so robust that there is no loss of quality when passing to general trace functions:

Theorem (KLMS)

Let φ be a $SL_3(\mathbb{Z})$-invariant cusp form. Let K be a trace function, one has

$$\sum_{n\sim q^{3/2}} \lambda_\varphi(1,n)K(n) \ll_{f,c(K)} q^{3/2-\delta+o(1)},\ \delta = 1/36.$$

Set

$$S(K,X) := \sum_{n\sim X} \lambda_\varphi(1,n)K(n).$$

If K is an additive character, S. Miller has proven an analog of Wilton's bound:

$$S(e_q(a.\bullet),X) \ll_f X^{3/4+o(1)}.$$

Wlog wma that K is not an additive character.
– The first step of the HN approach is to realize the function K within a one-parameter family of q-periodic functions. Define

$$\widehat{K}(z,h) := \begin{cases} \widehat{K}(z)e_q(-h\overline{z}) & q \nmid z \\ \widehat{K}(0) & q \mid z \end{cases}$$

for $(z,h) \in \mathbb{Z}^2$ so that

$$K(n,h) := \frac{1}{q^{1/2}} \sum_{z\in\mathbb{F}_q^\times} \widehat{K}(z,h)e_q(-nz).$$

Taking $h = 0$ in the above

$$K(n,0) = K(n) - \frac{\widehat{K}(0)}{q^{1/2}}.$$

and, more generally, for any probability measure ϖ on $\mathbb{F}_q^\times$, we have

$$K_\varpi(n,0) = K(n) - \frac{\widehat{K}(0)}{q^{1/2}}.$$

where

$$K_\varpi(n,h) := \sum_{u \in \mathbb{F}_q^\times} \varpi(u) K(n, \overline{u}h)$$

It follows that

$$\begin{aligned} S(K,X) = & \sum_{u\in\mathbb{F}_q^\times} \varpi(u) \sum_{|h|\leq H} S(K(\bullet,\overline{u}h),X) \\ & - \sum_{u\in\mathbb{F}_q^\times} \varpi(u) \sum_{0<|h|\leq H} S(K(\bullet,\overline{u}h),X) + Err \\ = & \mathcal{F} - \mathcal{O} + Err. \end{aligned}$$

We take ω to be supported on the classes $u \equiv \overline{p}.l \,(\mathrm{mod}\, q)$ for pairs of primes $p \sim P,\ l \sim L$ with $P, L < q^{1/2}$.

Bounding $\mathcal{F}$

$$\mathcal{F} = \frac{\log P}{P/2}\frac{\log L}{L/2}\sum_{p,l}\sum_{|h|\leq H}\sum_{n\sim X}\lambda_\varphi(1,n)K(n,p\bar{l}h).$$

We apply Poisson on h getting for the h, n sums

$$\frac{H}{q^{1/2}}\sum_{|r|\leq q/H}\sum_{n\sim X}\lambda_\varphi(1,n)\widehat{K}(-p\overline{lr})e(\frac{\overline{lr}pn}{q})$$

and apply reciprocity

$$e(\frac{\overline{lr}pn}{q}) = e(-\frac{\overline{q}pn}{lr})e(\frac{pn}{qlr}) \approx e(-\frac{\overline{q}pn}{lr}),$$

for $XP = (1/2)q^2L/H$ or $H = q^2L/2XP$.

We use the automorphy of φ through Voronoi summation formula:

$$\sum_{n\sim X}\lambda_\varphi(1,n)e(-\frac{\overline{q}pn}{lr})$$
$$\approx \frac{X}{(Lq/H)^{3/2}}\sum_{n\ll(Lq/H)^3/X}\lambda_\varphi(n,1)Kl_2(\pm\overline{p}qn;lr)$$

We then Cauchy to smooth out n

$$\sum\sum_{p,l,n,r}\cdots\leq(\sum_{n,r}|\lambda_\varphi(n,1)|^2)^{1/2}(\sum_{n,r}|\sum_{p,l}\widehat{K}(-p\overline{lr})Kl_2(\pm\overline{p}qn;lr)|^2)^{1/2}$$

and apply Poisson on the resulting n-sum

$$\sum_{n\ll(Lq/H)^3/X}Kl_2(\pm\overline{p}_1qn;l_1r)Kl_2(\pm\overline{p}_2qn;l_2r)$$

and use the expression of the Fourier transform of the product of Kloosterman sums in terms of Ramanujan sums.

We obtain that for $L \leq P^4$

$$\mathcal{F} \ll q^{o(1)}\left(\frac{X^{3/2}P}{qL^{1/2}} + X^{3/4}(qPL)^{1/4}\right).$$

and to be non-trivial one need at least that $X \geq q^{1+\eta}$.

Remark

At this stage, the only information (due to Deligne) we have used is that for K not an additive character, one has

$$\|\widehat{K}\|_\infty \ll_{c(K)} 1.$$

Bounding $\mathcal{O}$

Recall that

$$\mathcal{O} = \frac{\log P}{P/2}\frac{\log L}{L/2}\sum_{p,l}\sum_{\substack{0<|h|\leq H\\(h,l)=1}}\sum_{n\sim X}\lambda_{\varphi}(1,n)K(n,p\bar{l}h).$$

This time we immediately Cauchy to smooth n and evaluate

$$\sum_{\substack{p_1,h_1,l_1 \\ p_2,h_2,l_2}} \sum_{n\sim X} K(n, p_1\bar{l}_1 h_1)\overline{K(n, p_2\bar{l}_2 h_2)}$$

$$= \sum_{x_1,x_2\in\mathbb{F}_q^\times} \nu(x_1)\nu(x_2) \sum_{n\sim X} K(n, x_1)\overline{K(n, x_2)}$$

Since $X \geq q^{1+\eta}$, only the zero contribution in the dual variable survives and the sum becomes

$$\frac{X}{q^{1/2}} \sum_{x_1,x_2\in\mathbb{F}_q^\times} \nu(x_1)\nu(x_2)\frac{1}{q^{1/2}} \sum_{u\in\mathbb{F}_q} K(u, x_1)\overline{K(u, x_2)}$$

$$= \frac{X}{q^{1/2}} \sum_{x_1,x_2\in\mathbb{F}_q^\times} \nu(x_1)\nu(x_2)\frac{1}{q^{1/2}} \sum_{u\in\mathbb{F}_q} \widehat{K}(u, x_1)\overline{\widehat{K}(u, x_2)}$$

Moreover

$$\frac{1}{q^{1/2}} \sum_{u \in \mathbb{F}_q} \widehat{K}(u, x_1) \overline{\widehat{K}(u, x_2)} = L(x_1 - x_2)$$

with

$$L(x) = \frac{1}{q^{1/2}} \sum_{u \in \mathbb{F}_q^\times} |\widehat{K}(u)|^2 e(-\frac{\overline{u}x}{q}) + \frac{1}{q^{1/2}} |\widehat{K}(0)|^2.$$

The second term is no problem.
For the first term, observe that if $|\widehat{K}(u)|^2 = 1$ a.e. (which is the case for $K = \chi$ treated by HN) the first term is a Ramanujan sum hence very small.

In general we have the following elementary:

Lemma

Given $\mu, \nu, L : \mathbb{F}_q \to \mathbb{C}$ *we have*

$$\sum_{x_1, x_2 \in \mathbb{F}_q} \nu(x_1)\nu(x_2)L(x_1 - x_2) \leq q^{1/2}\|\nu\|_2^2\|\widehat{L}\|_\infty.$$

which is proven by separating x_1, x_2 in $L(x_1 - x_2)$ using the inverse Fourier transform formula and Cauchying.

In the present case we have

$$\widehat{L}(u) = |\widehat{K}(0)|^2 \delta_{u \equiv 0 \,(\mathrm{mod}\, q)} + |\widehat{K}(\bar{u})|^2 \delta_{u \not\equiv 0 \,(\mathrm{mod}\, q)},$$

and (assuming $PHL < q$)

$$\begin{aligned} \|\nu\|_2^2 &= |\{(p_1, h_1, l_1, p_2, h_2, l_2),\ p_1 \bar{l}_1 h_1 \equiv p_2 \bar{l}_2 h_2 \,(\mathrm{mod}\, q)\}| \\ &= |\{(p_1, h_1, l_1, p_2, h_2, l_2),\ p_1 l_2 h_1 = p_2 l_1 h_2\}| = (PHL)^{1+o(1)}. \end{aligned}$$

This yields

$$\mathcal{O} \ll_f q^{o(1)} \|\widehat{K}\|_\infty \frac{qX^{1/2}}{P}.$$

Combining the $\mathcal{F}$ and $\mathcal{O}$ bounds we conclude.

Remark

The only information used is that K – not being an additive character – satisfies

$$\|\widehat{K}\|_\infty \ll_{C(K)} 1.$$

It is important to have X close to or below q.

- Being able to go below the distribution range $X_d = q$ for $K(n) = Kl_3(n; q)$ would make it possible to evaluate asymptotically the first moment

$$\sum_{\chi \,(\mathrm{mod}\, q)} L(\varphi.\chi, 1/2)$$

  and to obtain non-vanishing results for central values of twists: so far this is known only on average over suitable composite moduli $q_1 q_2$ (W. Luo).
- For φ non-cuspidal such non-vanishing results for twisted L-functions are known (Das-Khan, Petrow, Zacharias).

$\mathrm{GL}_2 \times \mathrm{GL}_3$

We have

$$X_c = q^3,\ \theta_6 = 2/7,\ X_d = q^{3-1/2}.$$

Theorem (LMS)

Let f be a $\mathrm{SL}_2(\mathbb{Z})$-cusp form and φ be a $\mathrm{SL}_3(\mathbb{Z})$ cups form and

$$\lambda(n) = \lambda_\varphi(1,n).\lambda_f(n)$$

If K is "good", one has

$$\sum_{n \sim X} \lambda_\varphi(1,n)\lambda_f(n)K(n)$$
$$\ll_{f,\varphi,c(K)} (qX)^{o(1)}(X^{3/4}q^{11/16} + X^{2/3}.q^{11/12} + X.q^{-1/8}).$$

- For $X = q^3$ the bound saves $q^{-1/16}$
- The bound is non trivial as long as $X \geq q^{3-1/4+\eta}$. So this does not improve θ_6.

$\mathrm{GL}_2 \times \mathrm{GL}_3$

The proof follows the proof by P. Sharma of the special case $K = \chi$ (equivalent to the subconvex bound for $L(\varphi \times f.\chi, 1/2)$):

Theorem (Sharma)

Let f be a $\mathrm{SL}_2(\mathbb{Z})$-cusp form and φ be a $\mathrm{SL}_3(\mathbb{Z})$ cups form . One has

$$\sum_{n \sim q^3} \lambda_\varphi(1,n)\lambda_f(n)\chi(n) \ll_{\varphi,f} q^{3-\delta+o(1)},\ \delta = 1/16.$$

Corollary (Sharma)

Let φ be a $SL_3(\mathbb{Z})$-invariant cusp form. One has,

$$\sum_{n \sim q^{3/2}} \lambda_\varphi(1,n)\chi(n) \ll_f q^{3/2-\delta+o(1)},\ \delta = 1/32.$$

Sharma's proof uses

1. The δ-symbol method to decompose
$$\delta_{m=n}\lambda(1,m)\lambda_f(n)\chi(n).$$
2. Conductor decreasing trick.
3. GL_2 and GL_3-Voronoi on n and m.
4. Cauchy to smooth out m.
5. Poisson (aka GL_1-Voronoi). In this case, some non-zero frequencies contribute.
6. Squareroot cancellation in multivariable exponential sums by "invoking" the Adolphson-Sperber non-degeneracy criterion.

Excepted for the very last step, the proof does not make much use of the fact that χ is a Dirichlet character.

- In the end the most complicated exponential sum one need to face is: for $(l, m, p) \in \mathbb{F}_q^\times$ some parameters (arising from amplification and δ-symbol methods)

$$Z_{\ell,m,p}(v) := \frac{1}{q^{1/2}} \sum_{a \,(\mathrm{mod}\, q)} K(a)\mathrm{Kl}_2(\overline{p}^2 ma; q)\mathrm{Kl}_2(\overline{p}^3 \ell \overline{v} a; q)$$

$$\mathcal{C}_{\ell,m,p,\ell',m',p'}(h; q) := \frac{1}{q^{1/2}} \sum_{v \in \mathbb{F}_q^\times} Z_{l,m,p}(v)\overline{Z_{l',m',p'}(v + \overline{pp'}h)}.$$

- It is at this stage that the hypothesis that K is "good" comes in and to describe what "good" means one needs to know a bit more about...

Trace functions

Given $(\ell, q) = 1$, choose an embedding $\iota : \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$.
The basic datum is a Galois representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{F}_q[T]}/\mathbb{F}_q(T)) \to \mathrm{GL}(V)$$

for V a finite dimensional $\overline{\mathbb{Q}_\ell}$-vector space.

- We assume that ρ is (ι-)pure of weight 0: the eigenvalues of the Frobenius at any unramified place of $\mathbb{F}_q(T)$ have absolute value 1.
- The trace function associated with ρ is the function

$$K_\rho : t \in \mathbb{F}_q \mapsto \mathrm{tr}(\mathrm{Frob}_t | V^{I_t}) \in \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}.$$

(here "t" denote the "place" of the function field associated with the polynomial $T - t$.) It follows from purity that

$$\|K_\rho\|_\infty \leq \dim V.$$

Trace functions

The datum of this Galois representation ρ is equivalent to the datum of an ℓ-adic sheaf $\mathcal{F} = \mathcal{F}_\rho$ with nice properties (a middle extension sheaf) and

$$K_\rho = K_{\mathcal{F}}$$

is the trace function attached to that sheaf and the representation ρ is called the *monodromy* of the sheaf $\mathcal{F}$.

- The *geometric* monodromy $\overline{\rho}$ is the restriction of ρ to the *geometric* Galois (sub)-group

  $$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{F}_q[T]}/\overline{\mathbb{F}_q}(T)) \to \mathrm{GL}(V)$$

  and the *geometric monodromy group* is the Zariski closure of its image in $\mathrm{GL}(V)$.

Trace functions

- The *conductor* $c(K)$ of K (more correctly of ρ or $\mathcal{F}$) is formed by agregating (ie. summing) the local and global invariants of $\overline{\rho}$:
  - The dimension $\dim V$,
  - the number of ramified places of $\overline{\rho}$,
  - the Swan conductor at these places $\mathrm{swan}_{\overline{t}}(\rho)$ (which is zero unless the representation is wildly ramified there).

The goodness criterion

Definition

The trace function K is "good" if the sheaf $\mathcal{F}$ associated to K **does not** satisfy any of these conditions

- For $\lambda \in \mathbb{F}_q^\times - \{1\}$ the geometric monodromy of $\mathcal{F}$ has some quotient isomorphic to $[\times\lambda]^*\mathcal{K}\mathcal{L}_2$.
- For some $\lambda \in \mathbb{F}_q^\times - \{1\}$, $\mathcal{F}$ and $[\times\lambda]^*\mathcal{F}$ are geometrically isomorphic.
- The local monodromy of $\mathcal{F}$ at ∞ has a slope equal to $1/2$.

The goodness criterion

Recall that

$$Z_{\ell,m,p}(v) := \frac{1}{q^{1/2}} \sum_{a \pmod q} K(a)\mathrm{Kl}_2(\overline{p}^2 ma; q)\mathrm{Kl}_2(\overline{p}^3 \ell \overline{v} a; q)$$

$$\mathcal{C}_{\ell,m,p,\ell',m',p'}(h; q) := \frac{1}{q^{1/2}} \sum_{v \in \mathbb{F}_q^\times} Z_{l,m,p}(v)\overline{Z_{l',m',p'}(v + \overline{pp'}h)}.$$

Theorem

If K is "good" then whenever $h \neq 0 \pmod q$ or $(l, m, p) \neq (l', m', p')$ one has

$$\mathcal{C}_{\ell,m,p,\ell',m',p'}(h; q) \ll 1$$

The goodness criterion: idea of the proof

- The Z function can be obtained from K by a sequence of simple transformations (we assume $\ell = m = p = 1$ for simplicity)

$$K(x) \xrightarrow{\times \mathrm{Kl}_2} L(x) = K(x)\mathrm{Kl}_2(x) \xrightarrow{FT} \widehat{L}(y)$$

$$\xrightarrow{\mathrm{inv}} M(y) := \widehat{L}(y^{-1}) \xrightarrow{FT} \widehat{M}(u) \xrightarrow{\mathrm{inv}} M(u^{-1})$$

  where FT denote the Fourier transform and $\mathrm{inv} : x \to x^{-1}$ the inversion.
- These transformations have geometric analog at the level of sheaves and one can track how the singularities of $\mathcal{F}$ evolve when applying these (the deep but explicit work of Laumon on the local Fourier transform is used there) to see when the two copies of Z correlate.

The goodness criterion

The goodness criterion is pretty generic:

- $n \mapsto e_q(P(n))$ is good if $\deg P \geq 3$ and is not a monomial.
- The Kloosterman sum $n \mapsto \mathrm{Kl}_d(n)$ is good if $d \geq 3$.

There are however some notable "bad apples":

The goodness criterion

- $n \mapsto \chi(n)$ (Sharma's case): in that case the sums $Z_{\ell,m,p}(v)$ and $\mathcal{C}_{\ell,m,p,\ell',m',p'}(h;q)$ simplify considerably and the bound

$$\mathcal{C}_{\ell,m,p,\ell',m',p'}(h;q) \ll 1$$

  is still valid for $h \neq 0$ (and easier) and when $h = 0$ the failure is localized along an explicit and small diagonal set of the remaining parameters and the final bound remains valid.
- $n \mapsto \mathrm{Kl}_2(n)$: here the bound really fails but one can get around with a trick.

The goodness criterion

- $n \mapsto \mathrm{Kl}_2(n)$: here the bound really fails but one can get around with a trick: we have

$$\mathrm{Kl}_2(n; q) = \frac{q^{1/2}}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \chi(n)\overline{\varepsilon(\chi)^2}$$

so that

$$S(\mathrm{Kl}_2, q^3) = \frac{q^{1/2}}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \overline{\varepsilon(\chi)^2} S(\chi, q^3)$$

and applying the functional equation one find that

$$S(\mathrm{Kl}_2, q^3) \approx \frac{q^{1/2}}{\varphi(q)} \sum_{\chi \,(\mathrm{mod}\, q)} \varepsilon(\chi)^{6-2} S(\overline{\chi}, q^3) \approx S(\mathrm{Kl}_4, q^3)$$

and Kl_4 is good !

Thank you !