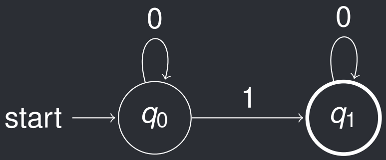
Definability on the Reals from Büchi Automata

A look at expansions of $(\mathbb{R}, <, +)$ by r-regular subsets



Alexi Block Gorman

The Fields Institute

Thematic Program on Trends in Pure & Applied Model Theory

Formal definition

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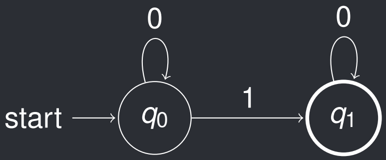

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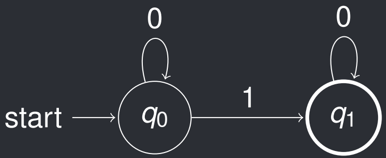

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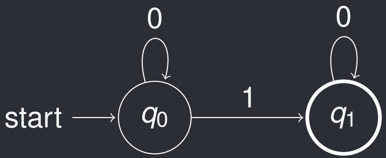

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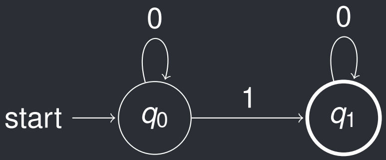

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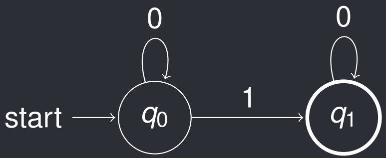

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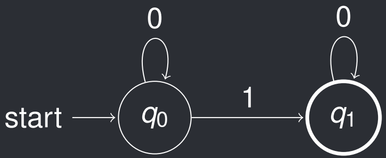

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- q_0 is the initial state,
- F is the set of accept states (denoted by **bold** circles or double circles).

Intro to automata

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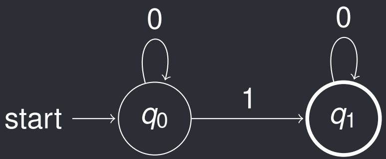

Example: The language this automaton recognizes is

$$L = 0^*10^* = \{0^n10^m : n, m \in \mathbb{N}\}.$$

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The class of regular expressions (e.g. $0^*10^* \cup (00, 11)^*$) generated from Σ via union, concatenation, and Kleene star is equivalent to the class of regular languages.

Finite automata in model theory

- In [6], Moosa and Scanlon define "*F*-sets" in order to unify the isotrivial case of the Mordell-Lang theorem in characteristic p (treated as exceptional by Hrushovski) with the non-isotrivial cases.

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- In [1], Bell and Moosa show that these F-sets are recognized by finite automata in a particular sense.
- To chracterize what these F-sets look like in terms of automata, they introduce the notion of *sparseness*: Say $L \subseteq \Sigma^*$ is sparse if $\#L(n) = |\{w \in L : |w| \leq n\}|$ grows at most polynomially in n.

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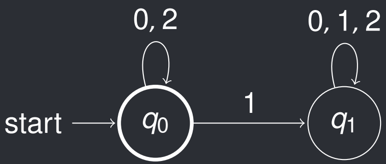

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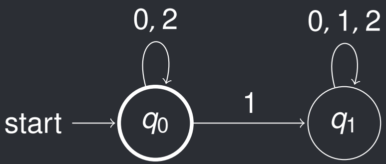

The input strings for this automaton are the ternary representations for points in $[0, 1]$, i.e. if $x = d_1 \frac{1}{3} + d_2 \frac{1}{9} + \ldots$ (with digits $d_1, d_2, \ldots \in \{0, 1, 2\}$) then "$d_1 d_2 \ldots$" is the input.

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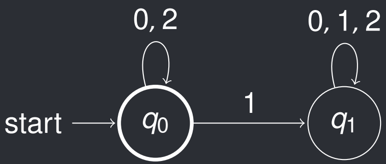

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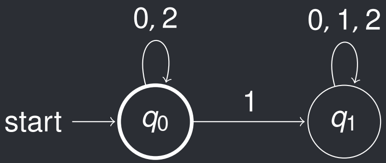
Say that $X \subseteq [0, 1]$ is *r*-regular if there is a BA that accepts an input iff the input is a base-r expansion of some $x \in X$.

What set does this BA recognize?



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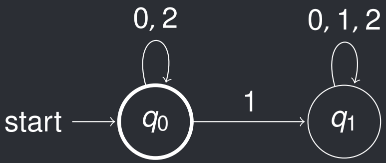

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This automaton accepts $x \in [0, 1]$ precisely if the ternary expansion of that point has either 0 or 2 as the coefficient for $\frac{1}{3^n}$ for each $n \in \mathbb{N}$.

ω-languages

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Theorem (Büchi, '62)

For every $L \subseteq \Sigma^\omega$ recognized by a Büchi automaton there are regular languages $V_1, \ldots, V_k, W_1, \ldots, W_k$ such that

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For the Cantor set, $\mathscr{C} = \{0, 2\}^\omega$

For $\mathbb{D} := \left\{ \frac{m}{2^n} : m, n \in \mathbb{N}, m \leq 2^n \right\}, \mathbb{D} = \{0, 1\}^* 0^\omega$

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Definition

Let $V_r(x, u, k)$ be a ternary predicate on $\mathbb{R}$ that holds precisely if $u = r^{-n}$ for some $n \in \mathbb{N}_{>0}$ and the n^{th} digit of the base-r representation of x is k.

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Corollary

The theory of $(\mathbb{R}, <, 0, +, V_r)$ is decidable.

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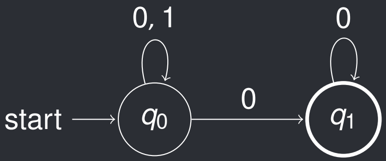

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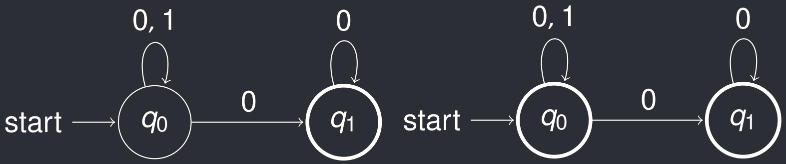

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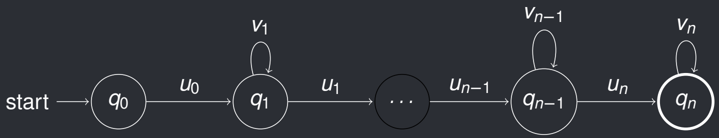

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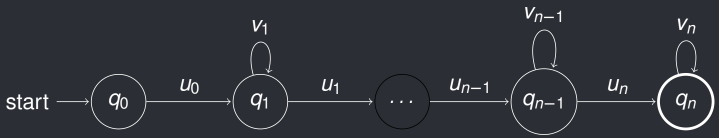

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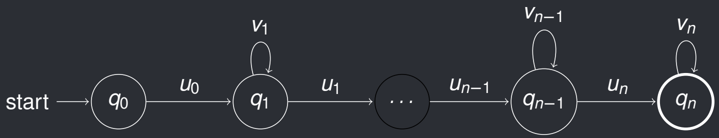

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If $X \subseteq [0, 1]$ *is* r*-sparse, the structures* $(\mathbb{R}, <, +, 0, r^{-\mathbb{N}})$ *and* $(\mathbb{R}, <, +, 0, 1, X)$ *define the same sets.*

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Corollary

If $X \subseteq [0, 1]$ *is* r*-sparse, the structure* $(\mathbb{R}, <, +, 0, 1, X)$ *is d-minimal and has NIP.*

Defining sparse sets from $r^{-\mathbb{N}}$

The key identity:

$$\frac{1}{1 - r^{-n}} = 1 + \frac{1}{r^n} + \frac{1}{r^{2n}} + \ldots = 1.0^n 1 0^n 1 \ldots$$

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$$.u_1 v_1^{n_1} \ldots u_{d-1} v_{d-1}^{n_{d-1}} u_d v_d v_d v_d v_d v_d v_d \ldots =$$

$$u_1 \times \frac{1}{r^{k_1}} + v_1 \times \left(\frac{r^{-(k_1+m_1)}}{1-r^{-m_1}} - \frac{r^{-k_1-m_1(n_1+1)}}{1-r^{-m_1}} \right) + \ldots$$

$$\ldots + u_d \times \frac{1}{r^{\sum k_i + \sum m_i(n_i+1)}} + v_d \times \frac{r^{-n_d - \sum k_i - \sum m_i(n_i+1)}}{1-r^{-n_d}}$$

Fractal dimension

Define Hausdorff s-measure of X as follows:

$$\mu_H^s(X) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{Diam}\, U_i)^s : \{U_i\}_i \text{ is a collection sets of diameter at most } \varepsilon \text{ covering } X \right\}$$

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$$\mu_H^s(X) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{Diam}\, U_i)^s : \{U_i\}_i \text{ is a collection sets of diameter at most } \varepsilon \text{ covering } X \right\}$$

The Hausdorff dimension of X is given by:
$d_H(X)$ is the unique $s \in \mathbb{R}$ such that

$$s' > s \implies \mu_H^{s'}(X) = 0 \ \& \ s' < s \implies \mu_H^{s'}(X) = \infty$$

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- $d_H(\mathbb{D}) = 0$.

Tameness dichotomy

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Lemma

A closed, r-regular subset of $[0, 1]$ *has positive Hausdorff dimension if and only if it is not r-sparse.*

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Call $C \subseteq \mathbb{R}$ a Cantor set if it is compact, has no isolated points, and has no interior.

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Theorem (B.G., Thesis)

If X is a closed r-regular subset of $[0,1]$ such that $0 < d_H(X) < 1$, then there is a unary set C definable in $(\mathbb{R}, <, +, 0, 1, X)$ such that C is a Cantor set.

Tameness dichotomy continued

In [5], Hieronymi & Walsberg show $(\mathbb{R}, <, +, C)$ interprets the theory of $(\mathscr{P}(\mathbb{N}), \mathbb{N}, \in, +1)$ if C is a Cantor set.

Tameness dichotomy continued

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Corollary

If $X \subseteq [0, 1]$ is an r-regular set such that $\overline{X}$ has no interior, then the following are equivalent:

1. $d_H(\overline{X}) = 0$
2. $(\mathbb{R}, <, +, 0, 1, X)$ *is d-minimal*
3. $(\mathbb{R}, <, +, 0, 1, X)$ *has NIP*
4. $(\mathbb{R}, <, +, 0, 1, X)$ *has* NTP_2

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Thank you!

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