NIP approximate subgroups

Workshop, Fields Institute

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- I will first discuss background and set the scene by looking at various contexts or "regimes".

Graph regularity I

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- almost all (V_i, W_j, R) are *regular* in the sense that sufficiently large subgraphs have almost the same density.
- The notions of small, almost all, sufficiently large, almost the same, depend on an (arbitrary) ϵ given in advance.
- More precisely: for all ϵ there is N_ϵ such that for all finite (V, W, R) there are partitions of V, W into at most N_ϵ pieces V_i, W_j, such that for all $(i, j) \notin \Sigma$ (where $|\cup_{(i,j)\in\Sigma} V_i \times W_j| \leq \epsilon |V \times W|$), (V_i, W_j, R) is "ϵ-regular".

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- In any case, under such restrictions one obtains better conclusions in the Szemeredi regularity lemma, with homogeneity (complete or empty) replacing regularity and sometimes with no exceptional set.
- Among the model theory references are Malliaris-Shelah and Chernikov-Starchenko. I have a largely expository paper in BSL 2020 "Domination and regularity".

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- The case of not necessarily abelian G is open.

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- For example, for any ϵ there is N_ϵ such that for every finite (G, A) with $x \cdot y \in A$, d-stable, there is a (normal) subgroup H of G of index at most N_ϵ such that A is equal to a union of cosets of H, up to a set of size $\leq \epsilon|H|$. In particular we DO obtain (*), in a strong form.

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- The more general d-NIP case is more complicated, with H replaced by a so-called "Bohr neighbourhood" (as well as existence of an exceptional set).
- However in all the conclusions, nothing non vacuous is said about A unless A is sufficiently large, for example $|A| \geq \epsilon|G|$.

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- It is pretty clear that nothing much can be said, even under stable or NIP hypotheses on A, other than A being a finite subset of G.
- So additional hypotheses have to be made, and one natural one (coming out of a line of research in additive combinatorics) is that of an *approximate subgroup* A of a group G.
- We define a subset A of a group G to be a k-approximate subgroup, if A is symmetric ($A = A^{-1}$) and $A \cdot A$ (the set of products of pairs of elements from A) is covered by k (left) translates of A.

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- Finally Breuillard, Green, Tao, extend this to not necessarily abelian groups, where the notion of a "coset nilprogession" replaces coset progression. More later.

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- In the rest of the talk, we give definitions and the precise theorem, and if we have time, discuss the proof and methods.

NIP

- Let M be a structure in language L and $\phi(x, y)$ an L-formula (where x, y could be tuples). Then $\phi(x, y)$ is d-NIP (with respect to M) if there do not exist a_i, and b_σ in M, for $i \in [d]$ and $\sigma \subseteq [d]$ such that $M \models \phi(a_i, b_\sigma)$ iff $i \in \sigma$ (for all i, σ).

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- Likewise we have the stronger properties of d-stability of a formula $\phi(x, y)$ and d-stability of a subset A of a group G.
- As mentioned earlier, given a finite bipartite graph $\mathcal{G}$ there is some d such that for any graph (V, W, R) which omits $\mathcal{G}$, the formula $(x, y) \in R$ is d-NIP.

Coset nilprogressions I

- Let G be an abelian group. By a *generalised arithmetic progression* of rank d in G we mean the image of a homomorphism from some d-dimensional box $\prod_{i=1,..,d}[-L_i, L_i] \subset \mathbb{Z}^d$ under a homomorphism $\pi : \mathbb{Z}^d \to G$. (Such an object is a 2^d-approxomate subgroup of G).

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- Properness means π is $1-1$. And to take account of finite subgroups, a *coset progression* (of rank d) is something of the form $P_0 + H$ where P_0 is a generalized arithmetic progression and H is a finite subgroup of G.
- The notion of a coset nilprogression is a certain nonabelian generalization: in place of the box above, one considers a "box" in the free nilpotent group of step r and rank k, consisting of elements which can be written as a word in the generators $e_1, ..., e_k$ and their inverses and where e_i and its inverse appear at most L_i times.

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- A *coset nilprogression* in G is a set P of the form P_0H where P_0 is a nilprogression and H is a finite subgroup of G normalized by P_0.

Statement of Theorem

Theorem 0.1

Suppose A is a finite subset of a group G, and A has k-tripling and d-NIP. Given $\epsilon > 0$, there is a coset nilprogression $P \subseteq G$, and a subset $Z \subseteq AP$ with $|Z| < \epsilon|A|$ (the error set) such that
(i) $P \subseteq AA^{-1} \cap A^{-1}A$ and $A \subseteq CP$ for some $C \subseteq A$,
(ii) For some $D \subseteq C$, $|(A \Delta DP) \setminus Z| < \epsilon|P|$.
(iii) For $g \in G \setminus Z$, $|gP \cap A| < \epsilon|P|$ or $|gP \cap A| > (1-\epsilon)|P|$.
Moreover rank and step and normal form of P, and the cardinality of C, are bounded by constants depending only on d, k, ϵ. And if G is abelian we can take P to be a (proper) coset progression.

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Moreover rank and step and normal form of P, and the cardinality of C, are bounded by constants depending only on d, k, ϵ. And if G is abelian we can take P to be a (proper) coset progression.

- Note the k-tripling hypothesis, which is slightly weaker than the approximate subgroup hypothesis, but essentially the same under NIP.

Comments on the statement

- In the main result of BGT, there is no NIP hypothesis, and no ϵ, and in the conclusion no Z, and in (i) we have the weaker statement $P \subseteq (A \cup A^{-1})^8$ and (ii), (iii) are replaced by the "bounded commensurability" condition that A is covered by C_k translates of P.

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- The conclusions (ii), (iii) in Theorem 0.1 are the typical "tame arithmetic regularity" conditions, that outside a small exceptional set A is essentially a bounded union of highly structured objects: in this case translates of a coset nilprogression.
- When G has exponent at most r, the coset nilprogession can be replaced by a subgroup H of index depending on the data, such that after throwing away the exceptional set Z A is a union of cosets of H, up to a set of size at most $\epsilon|H|$.

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- The use of model theory or logic has two aspects: (a) proving the relevant statement in the nonstandard (pseudofinite) environment and (b) pulling it down suitably to obtain the theorem.
- Part (b) is essentially routine. Part (a) is the main thing although the current proof still involves going down here and there and appealing to BGT. In any case, from here on it is model theory.

Comments on the proof II

- Let H be the subgroup of G generated by A, a so-called $\bigvee$-definable group, being a union of the definable sets $A^{\pm m} = A^m \cup A^{-m} \cup \{1\}$ (each of which is covered by finiteky many translates of the approximate group $A^{\pm 1}$).

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- H/Γ with the logic topology is a locally compact group $\mathbb{G}$, and let $\pi : H \to \mathbb{G}$ be the canonical surjective homomorphism.
- The main use of the NIP assumption is a "generic locally compact domination" theorem:

Comments on proofs III

- Let λ be Haar measure on $\mathbb{G}$, let X be a set in $\mathcal{R}$. Then there is a closed set $E \subset \mathbb{G}$ of λ-measure 0 such that for all $c \in \mathbb{G} \setminus Z$, exactly one of $\pi^{-1}(c) \cap X$, $\pi^{-1}(c) \setminus X$ is "μ-wide" (is not contained in a definable set of μ-measure 0).

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- This domination statement, plus several more steps, including appealing to (ultra) BGT, yield the following nonstandard version of Theorem 0.1 which suffices:
- For any $\epsilon > 0$ there is an internal coset nilprogression P in normal form, and $Z \subseteq AP$ with $Z \in \mathcal{R}$ and $\mu(Z) < \epsilon$, such that $P \subseteq AA^{-1} \cap A^{-1}A$, A is covered by finitely many translates of P, for each $g \in G \setminus Z$, $\mu(gP \cap A) = 0$ or $\mu(gP \setminus A) = 0$. And $A \setminus Z$ is a finite union of translates gP of P, up to measure 0.

Additional remarks

- In the case where A is d-stable, then in Theorem 0.1 there is NO exceptional set Z, and P is replaced by a (finite) subgroup of G.
- Why nilprogression in Theorem 0.1 and the nonstandard version above? Because, BGT prove that the connected component of H/Γ is an inverse limit of connected nilpotent Lie groups.
- (And when H is "definable" then the connected component of the compact group H/Γ is commutative.)
- ALSO, there is a model-theoretic measure-theoretic statement, involving generically stable ϕ-measures and assuming residual nilpotence of the connected component of the relevant H/Γ.