Enriched Grothendieck ring of a structure

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Definitions

Definition

Let M be a structure. $\mathrm{Def}(M)$ is the set of definable subsets of $\bigcup_{n\in\mathbb{N}} M^n$.

Note: "definable" means "definable with parameters".

First quotient

If $A, B \in \mathrm{Def}(M)$, $A \sim B$ if and only if, A is in definable bijection with B.
$[A]_1 =$ equivalence class of A.

Laws of the future ring

Law $+$

$[A]_1 + [B]_1 = [A' \cup B']_1$ where $[A]_1 = [A']_1$, $[B]_1 = [B']_1$ and $A' \cap B' = \emptyset$.
Its neutral element is $[\emptyset]_1$.

Law $\times$

$[A]_1 \times [B]_1 = [A \times B]_1$.
Its neutral element is $[\{*\}]_1$.

The Grothendieck Ring

$(\mathrm{Def}(M)/\sim, +, \times)$ is a semi-ring but is not a ring:
$[A]_1 + [C]_1 = [B]_1 + [C]_1$ does not imply $[A]_1 = [B]_1$.

Example

$[1]_1$ doesn't have an inverse for $+$.

Second quotient

If $a, b \in \mathrm{Def}(M)/\sim$, $a \simeq b$ if and only if $\exists c \in \mathrm{Def}(M)/\sim$ such that $a + c = b + c$.

Definition

There exists a unique (up to isomorphism) minimal ring that embeds $(\mathrm{Def}(M)/\sim)/\simeq$: **the Grothendieck ring** of M.

Notation

The Grothendieck ring of M is denoted by $\mathrm{K}_0(M)$.
We denote by $[A]$ the equivalence class of each $A \in \mathrm{Def}(M)$.

$$[A] = [B] \Leftrightarrow \exists C \in Def(M),$$
$$\begin{cases} A \cap C = B \cap C = \emptyset \\ A \sqcup C \text{ and } B \sqcup C \text{ are in definable bijection.} \end{cases}$$

Examples

Finite structures

$K_0(M) \cong \mathbb{Z}$, $\mathbb{N} = \{x \in K_0(M) | \exists A \in Def(M), [A] = x\}$.

Trivial structure

Let M be an infinite set. In the trivial language $\{=\}$,
$K_0(M) = \mathbb{Z}[X]$.

Onto-Php

Theorem (J. Krajíček, T. Scanlon, '00)

$$\mathrm{K}_0(M) \text{ is non trivial}$$
$$\Longleftrightarrow$$
$$\exists A \in Def(M) \text{ in definable bijection with } A \setminus \{*\}.$$

Proof

- $\Longleftarrow$: If A is in definable bijection with $A \setminus \{a\}$ ($a \in A$), then $[A \setminus \{a\}] = [A]$ and $[\{a\}] = 0$.
- $\Longrightarrow$: $0 = 1$ means going back to $\mathrm{Def}(M)$ that there exists $A \in \mathrm{Def}(M)$ such that $[A] + 1 = [A]$.

$\mathbb{Q}_p$ (L. van den Dries, '03)

$\mathrm{K}_0(\mathbb{Q}_p, \mathscr{L}_{\mathrm{ring}}) = \{0\}$

Formal Laurent series over finite fields (R. Cluckers, D. Haskell, '03)

$K_0(\mathbb{F}_q((t))) = \{0\}$ in $L_{ring} \cup \{t\}$.

Real closed field

$K_0(\mathbb{R}) \cong \mathbb{Z}$. (J. Krajíček, T. Scanlon, '00)

Proof

Let χ Euler characteristic on $\mathbb{R}$ valued in $\mathbb{Z}$.
Let A, B such that $\chi(A) = \chi(B)$.

Is there C disjoint from A and B such that $A \sqcup C$ and $B \sqcup C$ are in definable bijection ?

Proof continued

Proof

X and Y in definable bijection $\Leftrightarrow \begin{cases} \dim(A) = \dim(B) \\ \chi(A) = \chi(B). \end{cases}$

$\forall C$ such that $\Leftrightarrow \begin{cases} \dim(C) > \max(\dim(A), \dim(B)) \\ A \cap C = B \cap C = \emptyset \end{cases}$

$A \sqcup C$ is in definable bijection with $B \sqcup C$.

- $\mathbb{R} = \left]-\infty, 0\right[\ \sqcup \{0\} \sqcup \left]0, +\infty\right[$
- $[\mathbb{R}] = [\left]-\infty, 0\right[]$,
- $[\mathbb{R}] = [\left]0, +\infty\right[]$,
- $\Rightarrow [\mathbb{R}] = 2[\mathbb{R}] + 1 \Rightarrow [\mathbb{R}] = -1.$

Remark

$\forall k \in \mathbb{Z}, \exists A \in Def(\mathbb{R}), [A] = k.$

Proof

$k > 0$: A finite set of cardinaliy k.
$k < 0$: $[\mathbb{R} \times F]$ where $|F| = -k$.

- M finite structure $\Rightarrow$
  - $K_0(M) = \mathbb{Z}$,
  - $[M] = |M|$,
  - $\mathbb{N} = \{x \in K_0(M) | \exists A \in Def(M), [A] = x\}$.
- $\mathbb{R}$ real closed field $\Rightarrow$
  - $K_0(\mathbb{R}) = \mathbb{Z}$,
  - $[\mathbb{R}] = -1$,
  - $\mathbb{Z} = \{x \in K_0(M) | \exists A \in Def(\mathbb{R}), [A] = x\}$.

Enriched Grothendieck ring

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- Keep track of the set of equivalence classes of definable sets.

Enriched Grothendieck ring

Enriched Grothendieck ring

- Keep track of the set of equivalence classes of definable sets.
- Keep track of the class of the structure.

Enriched ring

Definition

An enriched ring is a triple (R, a, P) where

- R is a (unital, commutative) ring,
- $a \in R$,
- $P \subseteq R$ stable by addition and multiplication such that $P - P = R$, $0, 1, a \in P$.

P is the **set of positive elements**.
We say that (R, a, P) is an **enriched ring** of R.

Definition

Let M be a structure.
We can associate to it the enriched ring $(K_0(M), [M], P(M))$ where $P(M) = \{x \in K_0(M) | \exists A \in Def(M), [A] = x\}$.
It is called **enriched Grothendieck ring** of M.

In previous work,

Theorem

For every $I \subseteq \mathbb{Z}[X]$ ideal, there exists a structure M such that $K_0(M) = \mathbb{Z}[X]/I$.

Question

What are the enriched rings that are the enriched Grothendieck ring of a structure when $K_0(M) = \mathbb{Z}$ or $\mathbb{Z}/N\mathbb{Z}$?

Lemma

$(\mathbb{Z}, k, P)$ is an enriched Grothendieck ring

$$\Rightarrow \begin{cases} (\mathbb{Z}, k, P) = (\mathbb{Z}, k, \mathbb{Z}) \\ (\mathbb{Z}, k, P) = (\mathbb{Z}, k > 0, \mathbb{N}) \end{cases}$$

Proof

1. $\mathbb{N} \subseteq P$.
2. If $k < 0$ in P, then $\forall m, n \in \mathbb{N}, mk + n \in P$ (stability by addition).
3. $[M \setminus \{*\}] = [M] - 1 \Rightarrow [M] - 1 \in P$

Theorem

The converse is true: every enriched ring of the form $(\mathbb{Z}, k, \mathbb{Z})$ or $(\mathbb{Z}, k > 0, \mathbb{N})$ is the enriched Grothendieck ring of a structure.

Theorem

Every enriched ring of the form $(\mathbb{Z}/N\mathbb{Z}, k, \mathbb{Z}/N\mathbb{Z})$ is the enriched Grothendieck ring of a structure.

Case $K_0(M) = \mathbb{Z}$

1. **Case** $P = \mathbb{N}$**:** Let M be a finite structure of cardinality k.
   $\Rightarrow K_0(M) = \mathbb{Z}, [M] = k, P(M) = \mathbb{N}$.
2. **Case** $P = \mathbb{Z}$**:** For $\mathbb{R}$: $K_0(\mathbb{R}) = \mathbb{Z}, [\mathbb{R}] = -1, P(\mathbb{R}) = \mathbb{Z}$.
   F finite set of cardinality k.
   $\mathbb{R} \times F$ admits $(\mathbb{Z}, -k, \mathbb{Z})$ as enriched Grothendieck ring ?
   $\mathbb{R} \sqcup F$ admits $(\mathbb{Z}, k-1, \mathbb{Z})$ as enriched Grothendieck ring ?

Theorem

There exists a structure M with $K_0(M) = \mathbb{Z}/N\mathbb{Z}$ (and $[M] = 0$).

In the case where $K_0(M) = \mathbb{Z}/N\mathbb{Z}$, we want to prove that every $(\mathbb{Z}/N\mathbb{Z}, k, \mathbb{Z}/N\mathbb{Z})$ is an enriched Grothendieck ring

Disjoint union of structures

L and L' languages. $L_u = L \sqcup L'$.
M a L-structure and M' a L'-structure.
We want to make $M \sqcup M'$ a L_u-structure.

Constant

If $c \in L$ is a constant symbol, then c is interpreted in $M \sqcup M'$ as the constant c in the L-structure M.

Functions

If $f \in L$ is a function symbol of arity n, then f is interpreted

- on M^n as the function f in the L-structure M,
- on the complementary of M^n as a constant function equal to c' where $c' \in M'$.

Relations

If $R \in L$ is a relation symbol of arity n, then R is interpreted

- on M^n as the relation R in the L-structure M,
- on the complementary of M^n as the multi-diagonal: if $(x_1, \ldots, x_n) \in (M \sqcup M')^n \setminus M^n$, then $M \sqcup M' \models R(x_1, \ldots, x_n) \Leftrightarrow x_1 = \ldots = x_n$.

Definition

Let M be an L-structure and M' be an L'-structure. Let $M \sqcup M'$ be the disjoint union of M and M' considered as an $L \sqcup L'$-structure. We say that the $L \sqcup L'$-structure $M \sqcup M'$ is the **disjoint union** of M and M'.

Definable sets

Lemma

Let $M \sqcup M'$ be the disjoint union of M and M'.
Any definable set of $M \sqcup M'$ is a (finite) Boolean combination of sets of the form

- A with $A \in Def(M)$,
- B with $B \in Def(M')$,
- $\Pi_{i=1}^{n} A_i$ where $A_i \in Def(M) \cup Def(M')$.

Proposition

Let $M \sqcup M'$ be the disjoint union of two structures. Then $K_0(M \sqcup M')$ is equal to $K_0(M) \otimes_{\mathbb{Z}} K_0(M')$.

Lemma

- $i_1 : K_0(M) \hookrightarrow K_0(M \sqcup M')$, $i_2 : K_0(M') \hookrightarrow K_0(M \sqcup M')$
- $[M \sqcup M'] = i_1([M]) + i_2([M'])$,
- $P(M \sqcup M') = i_1(P(M)) + i_2(P(M'))$.

Corollary

Let M be an structure whose enriched Grothendieck ring is $(K_0(M), [M], P(M))$.
Let F be a finite set.
Then the enriched Grothendieck ring of $M \sqcup F$ is $(K_0(M), [M] + |F|, P(M))$.

Proposition

Let $k > 0$ and F of cardinality $k + 1$.
Then the enriched Grothendieck ring of $\mathbb{R} \sqcup F$ is $(\mathbb{Z}, k, \mathbb{Z})$.

Proposition

Let $k < 0$.
Then $(\mathbb{Z}, k, \mathbb{Z})$ is the enriched Grothendieck ring of the disjoint union of $-k$ copies of $\mathbb{R}$.

Theorem

There exists a structure M whose enriched Grothendieck ring is $(\mathbb{Z}/N\mathbb{Z}, 0, \mathbb{Z}/N\mathbb{Z})$

Lemma

Every enriched ring of $\mathbb{Z}/N\mathbb{Z}$ is the enriched Grothendieck ring of a structure.

Thank you !