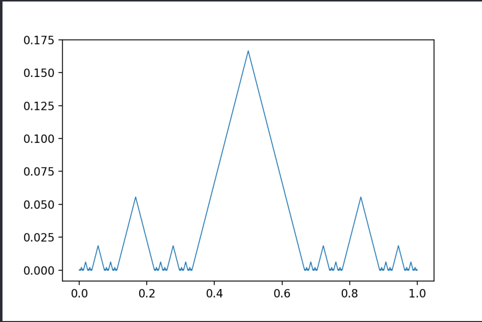
Fractal Dimensions and Büchi Automata

Dimension & definability of r-regular subsets of $\mathbb{R}$.

Joint work with Christian Schulz.

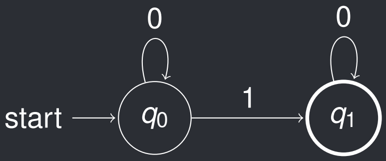


Alexi Block Gorman

The Fields Institute

Formal definition

An automaton is a special 5-tuple $(Q, \Sigma, \delta, q_0, F)$ such that:



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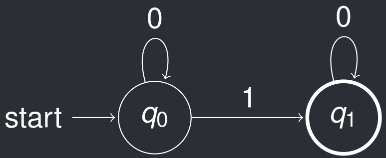

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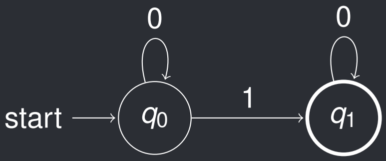

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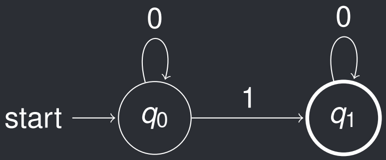

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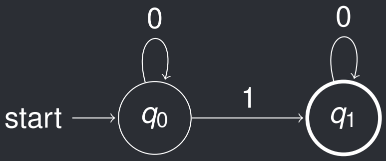

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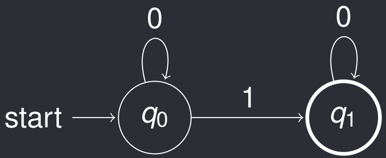

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- q_0 is the initial state,
- F is the set of accept states (denoted by **bold** circles or double circles).

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The class of regular expressions (e.g. $0^*10^* \cup (00, 11)^*$) generated from Σ via union, concatenation, and Kleene star is equivalent to the class of regular languages.

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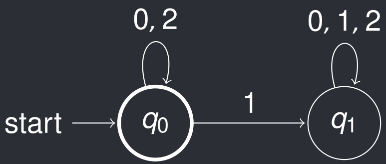

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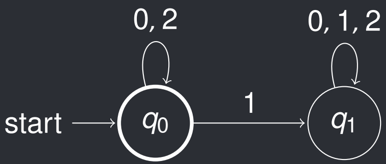

Interpret input strings for this automaton as ternary representations for points in $[0, 1]$, i.e. if $x = d_1 \frac{1}{3} + d_2 \frac{1}{9} + \ldots$ (with digits $d_1, d_2, \ldots \in \{0, 1, 2\}$) then "$d_1 d_2 \ldots$" is the input.

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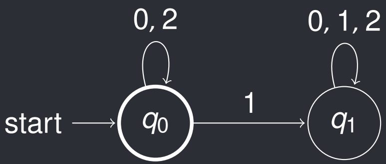

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Say that $X \subseteq [0, 1]$ is *r*-regular if there is a BA that accecepts an input iff the input is a base-r expansion of some $x \in X$.

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Theorem (Büchi, '62)

For every $L \subseteq \Sigma^\omega$ recognized by a Büchi automaton there are regular languages $V_1, \ldots, V_k, W_1, \ldots, W_k$ such that

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For $\mathbb{D} := \left\{ \frac{m}{2^n} : m, n \in \mathbb{N}, m \leq 2^n \right\}, \mathbb{D} = \{0, 1\}^* 0^\omega$

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Let $V_r(x, u, k)$ *be a ternary predicate on* $\mathbb{R}$ *that holds precisely if* $u = r^{-n}$ *for some* $n \in \mathbb{N}_{>0}$ *and the* n^{th} *digit of the base-r representation of* x *is* k*.*

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Corollary

The theory of $(\mathbb{R}, <, 0, +, V_r)$ is decidable.

Fractal dimensions

Let $X \subseteq \mathbb{R}^d$ be nonempty & bounded.

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- The Box-counting dimension of X is given by:

$$d_B(X) = \limsup_{\varepsilon \to 0} \frac{\log N(X, \varepsilon)}{\log \frac{1}{\varepsilon}} = \liminf_{\varepsilon \to 0} \frac{\log N(X, \varepsilon)}{\log \frac{1}{\varepsilon}}$$

where $N(X, \varepsilon)$ is the # of sets of the form

$$I_{\vec{z}} = [z_1 \varepsilon, (z_1 + 1)\varepsilon] \times \cdots \times [z_d \varepsilon, (z_d + 1)\varepsilon]$$

needed to cover X, where $\vec{z} = (z_1, \ldots, z_d)$ are integers.

Fractal dimensions continued

- Define Hausdorff s-measure of X as follows:

$$\mu_H^s(X) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{Diam}\, U_i)^s : \{U_i\}_i \text{ is a collection sets} \right.$$
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In general for $X \subseteq \mathbb{R}^n$ these dimensions satisfy the inequality $d_H(X) \leq d_B(X)$.

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$$d_H(\mathbb{D}) = 0 \quad \& \quad d_B(\mathbb{D}) = 1.$$

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Fact

Let X be an r-regular subset of $[0,1]^d$. Then:

$$d_B(X) = \frac{1}{\log r} h(X^{pre}).$$

If X is also closed, $d_H(X) = d_B(X)$ as well.

When dimensions disagree...

Definition

In a finite or Büchi automaton $\mathscr{A}$ with states indexed by $i \in I$, the cycle language $C_i(\mathscr{A}) \subseteq \Sigma^*$ contains all strings $w \in \Sigma^*$ such that there is a run of $\mathscr{A}$ from state i to itself via w.

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Let $\mathscr{A}$ with states indexed by $i \in I$ be a trim Büchi automaton, and let $\mathscr{A}$ recognize X. Let $X_i = C_i(\mathscr{A})^\omega$, and let F be the set of indices of accept states in $\mathscr{A}$. Then:

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(ii) $d_B(X) = \max_{i \in I} d_H(X_i)$.

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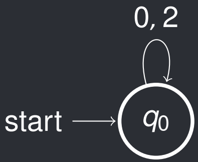

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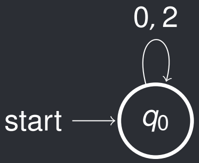

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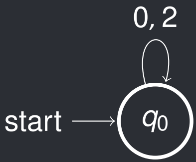

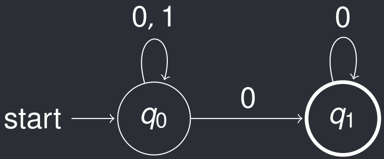

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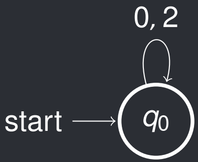

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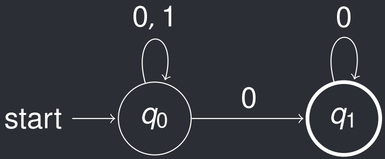

$$\max_{i \in F} d_H(X_i) = d_H(0^\omega) = 0$$

$$\max_{i \in I} d_H(X_i) = d_H(\Sigma^\omega) = 1$$

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Theorem (B. G., Schulz, 2021+)

Suppose $\mathcal{R}'$ is a reduct of $(\mathbb{R}, <, +, V_r)$. If there exists $A \subseteq [0, 1]$ definable in $\mathcal{R}'$ such that $d_H(A) \neq d_B(A)$, then either a Cantor set is definable in $\mathcal{R}'$, or some definable unary set is dense and codense in a subinterval of $[0, 1]$.

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Partial converse: If A is a Cantor set definable in $\mathscr{R}'$ then there exists a definable set A' such that $d_H(A') \neq d_B(A')$. Similarly if A is dense and codense in some interval $J \subseteq [0, 1]$ *and* the automaton that recognizes A is strongly connected.