On non-compact p-adic definable groups

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Section 1

The Peterzil-Steinhorn theorem in o-minimal groups

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Theorem (Peterzil-Steinhorn)

There is a definable subgroup $H \leq G$ such that

- $\dim(H) = 1$.
- *H is not definably compact.*
- *H is abelian.*
- *H is torsion-free.*

μ-types and μ-stabilizers

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Fact (Peterzil-Starchenko)

If p is a definable type, then $\mathrm{Stab}_\mu(p)$ is a definable subgroup of G.

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Peterzil and Starchenko later analyzed $\mathrm{Stab}_\mu(p)$ when $\dim(p) > 1$.

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Locally, $\text{Stab}_\mu(p)$ is the "limit" of the set $C - x$ as $x \to \infty$ along C.

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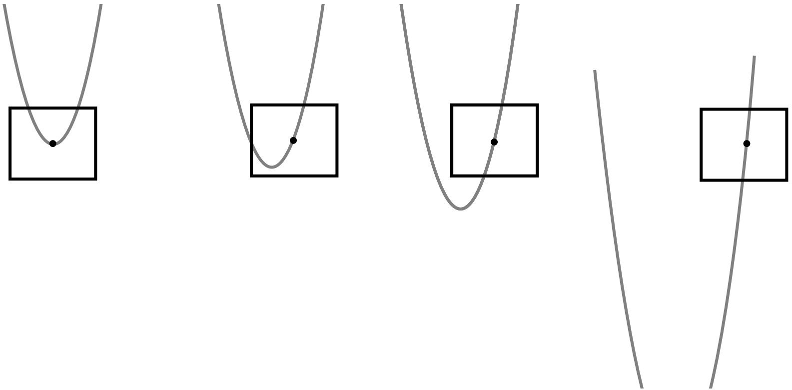

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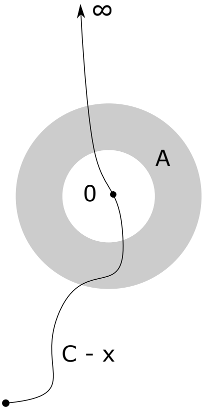
1. Draw an annulus A around $0 \in G$.



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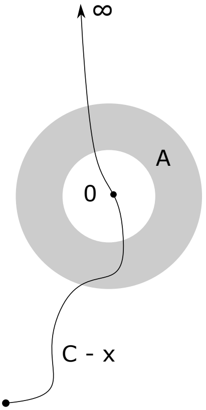

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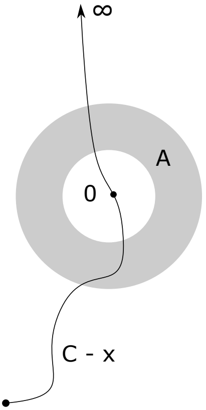

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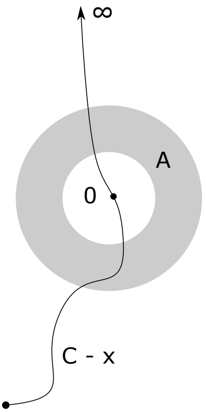

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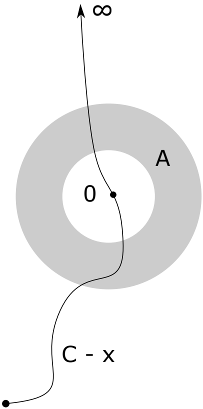

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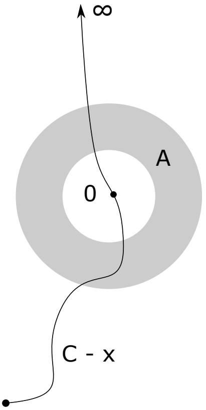

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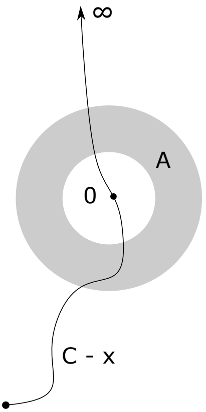

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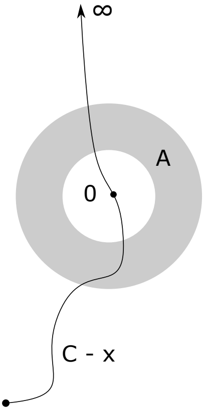
Connectedness plays a key role.



Section 2

p-adic analogues of the Peterzil-Steinhorn theorem

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Definition

G is **nearly abelian** if there is a definably compact definable subgroup $K \leq G$ with $[G, G] \subseteq K$.

Main results

Theorem (J., Yao)

If G is nearly abelian and not definably compact, then there is a 1-dimensional definable subgroup $H \leq G$ that is not definably compact.

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$\mathrm{Stab}_\mu(p)$ is a countable intersection of definable groups.

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Section 3

Definable compactness

Definable compactness in o-minimal definable manifolds

Work in an o-minimal structure $(M, <, +, \cdot, \ldots)$.

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- In general, this behaves like compactness. . .
  - A product of two definably compact sets is definably compact.
  - A continuous image of a definably compact set is definably compact.

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- It doesn't work in pCF either.

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A filtered intersection of a definable family of non-empty closed sets is non-empty.

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- $\mathbb{C}[[t]]$ is definably compact with its usual topology.

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Fact

In any setting, definable compactness satisfies the usual properties:

1. *Products of compact sets are compact.*
2. *Continuous images of compact sets are compact.*
3. *Closed subsets of compact sets are compact.*
4. *Compact sets in Hausdorff spaces are closed.*

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2. If U is a punctured neighborhood of 0 and $f : U \to X$ is definable and continuous, then $f(t)$ has a *cluster point* as $t \to 0$.
3. Every 1-dimensional definable type in X specializes to a point of X.

Section 4

Adapting Peterzil-Steinhorn to the p-adics

Back to the (p-adic) Peterzil-Steinhorn theorem

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Goal: understand $\mathrm{Stab}_\mu(p)$.

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   - **The principal difficulty**.

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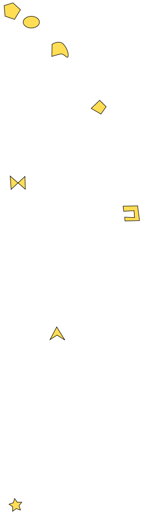

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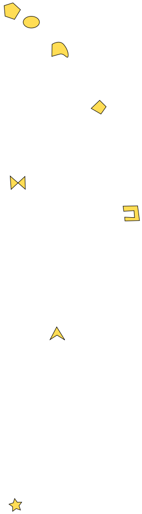

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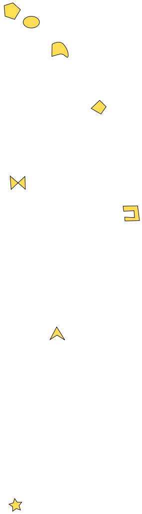

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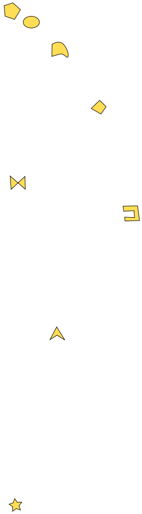

From large gaps to infinite dp-rank

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- M is NIP (Günaydın and Hieronymi)
- M does *not* have finite dp-rank.

Idea

If

$$X = \{1, 2, 3, 4, 5\}$$
$$Y = \{10, 20, 30, 40, 50\}$$
$$Z = \{100, 200, 300, 400, 500\}$$

then this map is injective:

$$X \times Y \times Z \to \mathbb{R}$$
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Why?

- Elements of Y are far apart relative to the size of elements of X.
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- If the X_i are definable and infinite, then

$$\text{dp-rk}(R) \geq \sum_{i=1}^{n} \text{dp-rk}(S_i) \geq n.$$

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5. n was arbitrary, so dp-rk$(M^*) \geq \aleph_0$.

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However, we must assume G is nearly abelian ($[G, G]$ is bounded).

Section 5

Towards the non-abelian case

What we expect

By mixing together...

1. The nearly abelian case,
2. Baldwin-Saxl for NIP groups,
3. Some representation theory and Lie theory,
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Note: we now abandon μ-stabilizers.

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- $\dim(A) > 0$ and A is non-compact abelian; use the abelian case.

Section 6

Towards dp-minimal fields

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New complications:

- Minor problem(?): no Skolem functions
- Major problem: definable compactness is broken

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Definable compactness in ACVF

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- $\mathcal{O}$ is *not* definably compact.

$$\varnothing = \bigcap_{\gamma > 0} \{x \in \mathcal{O} : 0 < v(x) < \gamma\}$$

- *No* infinite subset of K is definable compact.

Γ-compactness

Let (K, v) be a dp-minimal valued field with valuation group Γ.

Definition

A definable topological space X is Γ-**compact** if the following holds. Let $\{F_\gamma\}_{\gamma \in \Gamma}$ be a definable family of non-empty closed subsets of X. Suppose that

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Γ-compactness seems to have all the good properties.