Decidability and definability in unramified henselian valued fields

§0 Classical results

Notation: (K,v) valued field
Kv residue field
vK value group

Def: A valued field (K,v) of mixed characteristic $(0,p)$ is called unramified if $v(p)$ is the minimum positive element of vK.

Theorem (Ax-Kochen / Ershov) Let (K,v), (L,w) be two unramified henselian valued fields with perfect residue fields Kv, Lw.
Then:

$$\underbrace{(K,v) \equiv (L,w)}_{\text{in } \mathcal{L}_{val} = \mathcal{L}_{ring} \cup \{\mathcal{O}\}} \iff \underbrace{Kv \equiv Lw}_{\text{in } \mathcal{L}_{ring}} \text{ and } \underbrace{vK \equiv wL}_{\text{in } \mathcal{L}_{oag}}$$

Proof sketch:
"$\Rightarrow$" obvious
(res. field & value group are interpretable)

"$\Leftarrow$" Assume $Kv \equiv Lw$ and $vK \equiv wL$.

Taking ultraproducts of (K,v) and (L,w), we may assume:
(K,v), (L,w) $\aleph_1$-saturated
$Kv \cong Lw$, $vK \cong wL$

Let $\hat{v}, \hat{w}$ denote the finest proper coarsening of v resp. w

$$\begin{tikzcd} K \arrow[d, dash, "\hat{v}"] & & L \arrow[d, dash, "\hat{w}"] \\ K\hat{v} \arrow[d, dash, "\bar{v}"] & & L\hat{w} \arrow[d, dash, "\bar{w}"] \\ Kv & \cong & Lw \end{tikzcd}$$

Then: $(K\hat{v}, \bar{v})$ unramified, value group $\cong \mathbb{Z}$ with $\bar{v}(p) \mapsto 1$ and complete.

Fact (Hasse-Schmidt, Teichmüller, Witt)
Given a perfect field k of $\mathrm{char}(k) = p > 0$ there is a unramified complete discrete valued field $(W(k), v_k)$ with residue field k. $(W(k), v_k)$ is unique up to unique isom.

$$\begin{tikzcd} K \arrow[d, dash, "\hat{v}"] & & L \arrow[d, dash, "\hat{w}"] \\ K\hat{v} \arrow[d, dash, "\bar{v}"] & \cong & L\hat{w} \arrow[d, dash, "\bar{w}"] \\ Kv & \cong & Lw \end{tikzcd}$$

$\mathrm{char}(K, K\hat{v}) = \mathrm{char}(L, L\hat{w}) = (0,0)$, $K\hat{v} \cong L\hat{w}$, $\hat{v}K \cong \hat{w}L$

$AKE_{0,0} \Rightarrow (K, \hat{v}) \equiv (L, \hat{w})$

J. Robinson: (K,v) henselian, unramified. Then $\mathcal{O}_v$ is $\mathcal{L}_{ring}$-definable via the formula

$$\varphi(x) \equiv \exists y \, (y^2 = 1 + p x^2) \qquad \text{(for } p \neq 2)$$

$$\Rightarrow \quad (K,v) \equiv (L, w) \qquad \boxtimes$$

In a similar vein, one gets:

(K,v), (L,w) unramified henselian, Kv perfect

Ax-Kochen / Ershov

- $(K,v) \subseteq (L,w)$ and $Kv \preccurlyeq Lw$ and $vK \preccurlyeq wL$
  $\Rightarrow \quad (K,v) \preccurlyeq (L,w)$

Pas

- Kv is stably embedded in (K,v) as a pure field.

Bélair:

- Kv NIP $\Rightarrow$ (K,v) NIP

TODAY: How about imperfect residue fields?

jt work with S. Anscombe

§1 Cohen rings and imperfect residue fields

Fact (Hasse-Schmidt, Cohen)
Given a field k of $\mathrm{char}(k) = p > 0$ there is a unramified complete discrete valued field $(C(k), v_k)$ with residue field k, called a Cohen field over k.
Any two Cohen fields over k are isomorphic.

Corollary (AKE for imperfect residue fields)
Let (K,v) and (L,w) be two henselian unramified valued fields.
Then:

$$\underbrace{(K,v) \equiv (L,w)}_{\text{in } \mathcal{L}_{val} = \mathcal{L}_{ring} \cup \{\mathcal{O}\}} \iff \underbrace{Kv \equiv Lw}_{\text{in } \mathcal{L}_{ring}} \text{ and } \underbrace{vK \equiv wL}_{\text{in } \mathcal{L}_{oag}}$$

Note: $(C(k), v_k)$ is no longer rigid. So, in order to reprove the other classical results, we need to understand morphisms.

Def: Let (C,v) be a complete discrete unramified valued field with residue field k.
Let $\mathcal{B} \subseteq k$ be a p-basis. A map $s : \mathcal{B} \to C$ s.th. $\mathrm{res}(s(b)) = b$ for all $b \in \mathcal{B}$ is called a **map of representatives**.

Fact (Cohen, building on Teichmüller)
Assume $\mathrm{char}(k) = p > 0$, and let (C_1, v_1), (C_2, v_2) be Cohen fields over k.
Let $\beta \subseteq k$ be a p-basis and $s_i : \beta \to C_i$ be maps of representatives.
Then, there is a unique isom. of valued fields $\phi : C_1 \to C_2$ s.th. $s_1 \circ \phi = s_2$.

Consequences: (with some TLC)

(K,v), (L,w) unramified henselian

$AKE_{\preceq}$: $(K,v) \subseteq (L,w)$, $Kv \preceq Lw$, $vK \preceq wL$
$\Rightarrow (K,v) \preceq (L,w)$

Stable embeddedness:
Kv is stably embedded in (K,v) as a pure field.

NIP transfer:
Kv NIP $\Rightarrow$ (K,v) NIP

§2 Finitely ramified henselian valued fields

Def: (K,v) of mixed characteristic $(0,p)$ is **finitely ramified** if $(0, v(p)] \subseteq vK$ is finite.

Note: In order to understand $Th(K,v)$, we need to fix a degree of ramification $e \in \mathbb{N}_{>0}$ (i.e. $(0, v(p)]$ has e-many elements

Well-Known: $AKE_{\equiv}$ fails

e.g.

$\mathbb{Q}_3(\sqrt{3})$, $\mathbb{Q}_3(\sqrt{6})$ are distinct extensions of $\mathbb{Q}_p$ with $e=2$, residue field $\mathbb{F}_3$, value group $\mathbb{Z}$.

Example (Stable embeddedness of Kv fails)

[$p \neq 2$] $k = \mathbb{F}_p(\alpha_1, \alpha_2)$ with α_1, α_2 alg. indep.

$\rightsquigarrow$ ex. $\bar{\sigma} \in Aut(k)$ $\bar{\sigma}(\alpha_1) = \alpha_2$

(K,v) unram. hens., $Kv = k$, $a_1 \in K$ with $res(a_1) = \alpha_1$

any element with $res(a_2) = \alpha_2$

$L := K(\sqrt{pa_1})$

$\Rightarrow$ v extends uniquely to L, (L,v) is fin ramified, $Lv = k$, $[vL : vK] = 2$

Then: L contains no square root of pa_2

$\Rightarrow$ $\bar{\sigma}$ does not lift to $\sigma \in Aut(L)$

Since o/w, we claim that $\sigma(\sqrt{pa_1})$ is a root of pa_2 for some lift a_2 of α_2:

$$\mathrm{res}(\sigma(\sqrt{pa_1})^2/p) = \mathrm{res}(\sigma(a_1)) = \bar{\sigma}(\mathrm{res}(a_1))$$
$$= \bar{\sigma}(\alpha_1) = \alpha_2. \qquad \boxtimes$$

[This example works just as well with perfect residue field.]

We can prove nonetheless:

NIP: (K,v) finitely ramified, henselian Kv NIP $\Rightarrow$ (K,v) NIP

$AKE_{\preccurlyeq}$: $(K,v) \subseteq (L,w)$ finitely ramified of ramification degree $e>0$. Then

$Kv \preccurlyeq Lw$ and $vK \preccurlyeq wL \Rightarrow (K,v) \preccurlyeq (L,w)$.