Hilbert-type operator induced by radial weight

Elena de la Rosa

Departamento de Análisis Matemático...
Universidad de Málaga

Joint work with José Ángel Peláez

Focus Program on Analytic Function Spaces and their Applications, Fields Institute in Toronto, Canada.
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Notation

- $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.
- $\mathcal{H}(\mathbb{D}) = \{g : g$ is analytic in $\mathbb{D}\}$.
- For $0 < p \le \infty$, H^p is the classical **Hardy space**,

$$||f||_{H^p} = \sup_{0 \le r < 1} M_p(r,f) < \infty.$$

- The Bloch space $\mathcal{B}$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |f'(z)|(1-|z|) + |f(0)| < \infty.$$

- For an integrable function $\omega : [0,1) \to [0,\infty)$, the extension to $\mathbb{D}$ defined by $\omega(z) = \omega(|z|)$, $z \in \mathbb{D}$ is called a radial weight.

- For a radial weight ω, $0 < p < \infty$, let $L^p_{\omega,[0,1)}$ be the space of measurable functions such that $\|f\|^p_{L^p_{\omega,[0,1)}} = \int_0^1 |f(t)|^p \omega(t)\, dt < \infty$.

- For $0 < p < \infty$ and a weight ω, L^p_ω is the space of measurable functions f for which
$$\|f\|^p_{L^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty$$
where $dA(z) = \frac{dx\,dy}{\pi}$.

- The **weighted Bergman space** is $A^p_\omega = \mathscr{H}(\mathbb{D}) \cap L^p_\omega$. When $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$, $-1 < \alpha < \infty$ is a standard weight, this space is called standard Bergman space and is denoted by A^p_α.

- $\widehat{\omega}(r)=\int_r^1\omega(s)\,ds$. We assume that $\widehat{\omega}(r)>0$, $r\in(0,1)$, for otherwise $A^p_\omega=\mathcal{H}(\mathbb{D})$.
- We define the moments for $x\ge 0$, $\omega_x=\int_0^1 r^x\omega(r)dr$. ω_x is a decreasing function of x.
- If ω is a radial weight, $\|\cdot\|_{A^2_\omega}$-convergence implies uniform convergence on compact subsets of $\mathbb{D}$, so A^2_ω is a closed subspace of L^2_ω and the point evaluations $L_zf=f(z), z\in\mathbb{D}$, are bounded linear functionals on A^2_ω.
- There exist reproducing kernels $B^\omega_z\in A^2_\omega$ such that

$$f(z)=\langle f,B^\omega_z\rangle_{A^2_\omega}=\int_{\mathbb{D}}f(\zeta)\,\overline{B^\omega_z(\zeta)}\,\omega(\zeta)\,dA(\zeta),\quad f\in A^2_\omega.$$

Hilbert operator

Hilbert matrix is

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & . \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & . \\ . & . & . & . \end{pmatrix}$$

The Hilbert operator is defined by its action on the Taylor coefficients

$$H(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n.$$

Previous results

- H is not bounded in H^1 and H^∞ (Diamantopoulos-Siskakis(2000)).
- $H : H^p \to H^p$ if and only if $1 < p < \infty$ (Diamantopoulos-Siskakis(2000)).
- $H : A^p_\alpha \to A^p_\alpha$ if $1 < p$ and $-1 < \alpha < p-2$ (D(2004)),P-R(2013)).

An integral representation

$$H(f)(z) = \int_0^1 \frac{f(t)}{1-tz}\,dt$$

If $\omega = 1$, $B_\zeta^\omega(z) = \frac{1}{(1-z\overline{\zeta})^2}$ and

$$H(f)(z) = \int_0^1 f(t)\left(\frac{1}{z}\int_0^z \frac{1}{(1-t\varepsilon)^2}\,d\varepsilon\right)dt = \int_0^1 f(t)\left(\frac{1}{z}\int_0^z B_t^\omega(\varepsilon)\,d\varepsilon\right)\omega(t)\,dt.$$

For ω radial weight, we define the Hilbert-type operator induced by ω

$$H_\omega(f)(z) = \int_0^1 f(t)\left(\frac{1}{z}\int_0^z B_t^\omega(\varepsilon)d\varepsilon\right)\omega(t)dt = \int_0^1 f(t)K_t^\omega(z)\omega(t)dt,$$

where $K_t^\omega(z) = \frac{1}{z}\int_0^z B_t^\omega(\varepsilon)d\varepsilon$

Problem

Studying the action of the operator H_ω on Hardy and weighted Bergman spaces.

Some background on Bergman reproducing kernels

- If $\omega(z) = (\alpha+1)(1-|z|^2)^\alpha$ is an standard weight, then $B_\zeta^\omega(z) = \frac{1}{(1-z\overline{\zeta})^{2+\alpha}}$.
- **First obstacle:** In general, B_ζ^ω can have zeros and we don't have an explicit formula.
- We use $B_z^\omega(\zeta) = \sum_n e_n(\zeta)\overline{e_n(z)}$, $\quad z,\zeta \in \mathbb{D}$, $\{e_n\}$ is any orthonormal basis of A_ω^2.
- If ω is a radial weight, $\left\{\frac{z^j}{\sqrt{2\omega_{2j+1}}}\right\}$, $j \in \mathbb{N}\cup\{0\}$

$$B_z^\omega(\zeta) = \sum_{j=0}^\infty \frac{(\zeta\overline{z})^j}{2\omega_{2j+1}}.$$

Case $p = \infty$.

- H is not bounded in H^∞.
- If $x \in (0,1)$

$$H_\omega(1)(x) = \sum_{n=0}^{\infty} \frac{\omega_n}{2\omega_{2n+1}(n+1)} x^n \ge \frac{1}{2x} \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \frac{1}{2x} \log\left(\frac{1}{1-x}\right),$$

so H_ω is never bounded in H^∞.
- $H : H^\infty \to \mathscr{B}$ is bounded.

Question

Which are the radial weights ω such that $H_\omega : H^\infty \to \mathscr{B}$ is bounded?

Doubling weights of the class $\widehat{\mathcal{D}}$.

Theorem (Peláez-R. (2021))

Let ω be a radial weight. Then, $H_\omega : H^\infty \to \mathscr{B}$ is bounded if and only if there exists $C > 0$ such that

$$\widehat{\omega}(r) \le C\widehat{\omega}\left(\frac{1+r}{2}\right) \text{ for all } 0 \le r < 1 \quad (1)$$

$\omega \in \widehat{\mathcal{D}}$ if ω satisfies (1).

Examples

- Standard weights and $v_\beta(r) = \frac{1}{(1-r)(\log \frac{e}{1-r})^\beta}$, $\beta > 1$ belong to $\widehat{\mathcal{D}}$;
- $\omega(r) = \exp\left(-\frac{1}{1-r}\right) \notin \widehat{\mathcal{D}}$;

Doubling weights of the class $\widehat{\mathcal{D}}$.

$\widehat{\mathcal{D}}$ characterizes (Peláez-Rättyä (2019))

- Radial weights such that

$$|f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^p \omega(z)\, dA(z) \le C \|f\|^p_{A^p_\omega}, \quad f \in \mathcal{H}(\mathbb{D}),$$

- Radial weights such that the Bergman projection induced by ω

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{B^\omega_z(\zeta)} \omega(\zeta) dA(\zeta),$$

is bounded from L^∞ to $\mathcal{B}$.

Sketch of the proof

Theorem (Peláez-R. (2021))

Let ω be a radial weight. Then, $H_\omega : H^\infty \to \mathscr{B}$ is bounded if and only if $\omega \in \widehat{\mathscr{D}}$.

- $H_\omega(f)'(z) = \int_0^1 f(t) G^\omega(z,t)\omega(t)\,dt$, with $G^\omega(z,t) = \sum_{n=1}^{\infty} \frac{t^n z^{n-1}}{2\omega_{2n+1}} \frac{n}{n+1}$,
- $\omega \in \widehat{\mathscr{D}}$ if and only if for each $k > 1$ there exists $C > 0$ such that $\omega_n \le C\omega_{kn}$, $n \in \mathbb{N}$.

$$\begin{aligned}
\sup_{z\in\mathbb{D}}(1-|z|)|H_\omega(f)'(z)| &\le \|f\|_{H^\infty} \sup_{z\in\mathbb{D}}(1-|z|) \int_0^1 \left(\sum_{n=1}^{\infty} \frac{t^n |z|^{n-1}}{2\omega_{2n+1}} \frac{n}{n+1} \right) \omega(t)\,dt \\
&\le \|f\|_{H^\infty} \sup_{z\in\mathbb{D}}(1-|z|) \sum_{n=1}^{\infty} \frac{|z|^{n-1}\omega_n}{2\omega_{2n+1}}. \\
&\le C\|f\|_{H^\infty} \sup_{z\in\mathbb{D}}(1-|z|) \sum_{n=1}^{\infty} |z|^{n-1} \le C\|f\|_{H^\infty},
\end{aligned}$$

For each $N \in \mathbb{N}$

$$\begin{aligned}
\infty > \|H_\omega(1)\|_{\mathscr{B}} &\geq \sup_{z\in\mathbb{D}}(1-|z|)\left|\int_0^1 G^\omega(z,t)\omega(t)\,dt\right| \\
&\geq \sup_{x\in(0,1)}(1-x)\int_0^1 G^\omega(x,t)\omega(t)dt \\
&\geq C\sup_{x\in(0,1)}(1-x)\sum_{n=1}^{N+1} x^{n-1}\frac{\omega_n}{\omega_{2n+1}} \\
&\geq C\frac{1}{N}\sum_{n=E\left(\frac{3N}{4}\right)}^{N+1}\frac{\omega_n}{\omega_{2n+1}} \geq C\frac{\omega_{N+1}}{\omega_{2E\left(\frac{3N}{4}\right)+1}},
\end{aligned}$$

that is $\omega_{N+1} \leq C\omega_{2E\left(\frac{3N}{4}\right)+1}$, $N \in \mathbb{N}$.

Boundedness in H^1.

The classical Hilbert operator is not bounded on H^1 (Diamantopoulos-Siskakis (2000))

Theorem (Peláez-R. (2021))

Let $\omega \in \widehat{\mathcal{D}}$. Then, the following statements are equivalent:

(i) $H_\omega : H^1 \to H^1$ *is bounded;*

(ii) *The measure* $\mu_\omega(z) = \omega(z)\left(1 + \int_0^{|z|} \frac{ds}{\widehat{\omega}(s)}\right)\chi_{[0,1)}(z)\,dA(z)$ *is a* 1*-Carleson measure for* H^1*;*

(iii) ω *satisfies the condition*

$$\sup_{a\in[0,1)} \frac{1}{1-a}\int_a^1 \omega(t)\left(1 + \int_0^t \frac{ds}{\widehat{\omega}(s)}\right) dt < \infty.$$

Boundedness in H^p, $1 < p < \infty$

Problem: Obtaining a description of the radial (or doubling) weights such that $H_\omega : H^p \to H^p$ is bounded.

This problem is open.

$$\int_0^s |f(t)|^p \, dt \le \pi M_p^p(s,f), \quad 0 < p < \infty, \quad f \in \mathcal{H}(\mathbb{D}), \tag{2}$$

Problem: Describing the radial (or doubling) weights such that $H_\omega : L^p_{[0,1)} \to H^p$ is bounded.

Theorem (Peláez-R. (2021))

Let $\omega \in \widehat{\mathcal{D}}$ and $1 < p < \infty$. Then $H_\omega : L^p_{[0,1)} \to H^p$ is bounded if and only if ω satisfies

$$\sup_{0<r<1} \left(1 + \int_0^r \frac{1}{\widehat{\omega}(t)^p}\,dt\right)^{\frac{1}{p}} \left(\int_r^1 \omega(t)^{p'}\,dt\right)^{\frac{1}{p'}} < \infty. \tag{3}$$

In particular if (3) holds, $H_\omega : H^p \to H^p$ is bounded.

Boundedness in A^p_ν, $1 < p < \infty$, ν radial weight

The estimate

$$\int_0^1 |f(t)|^p \widehat{\nu}(t)\, dt \le \frac{\pi}{2} \|f\|^p_{A^p_\nu}, \quad f \in \mathscr{H}(\mathbb{D}),$$

yields $A^p_\nu \subset L^p_{\widehat{\nu},[0,1)}$ and the embedding is sharp.

Question

Obtaining a description of the doubling weights ω and the radial weights ν such that $H_\omega : L^p_{\widehat{\nu},[0,1)} \to A^p_\nu$ is bounded.

Theorem (Peláez-R. (2021))

Let $1 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and ν a radial weight on $\mathbb{D}$. Then, $H_\omega : L^p_{\widehat{\nu},[0,1)} \to A^p_\nu$ is bounded if and only if

$$\sup_{0<r<1} \left(1 + \int_0^r \frac{\widehat{\nu}(t)}{\widehat{\omega}(t)^p}\,dt\right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\omega(t)}{\widehat{\nu}(t)^{\frac{1}{p}}}\right)^{p'} dt\right)^{\frac{1}{p'}} < \infty. \tag{4}$$

In particular if (4) holds, $H_\omega : A^p_\nu \to A^p_\nu$ is bounded.

Sketch of the proof (Sufficiency)

1. This result plays a key role (Mateljević -Pavlović (1984), Flett (1972)): for all $1 < q < p < \infty$,

$$\|f\|_{H^p} \le C|f(0)|^p + C\int_0^1 M_q^p(t,f')(1-t)^{p\left(1-\frac{1}{q}\right)}dt, \quad f \in \mathcal{H}(\mathbb{D}).$$

2. Through integration, for all $1 < q < p < \infty$ and ν be a radial weight,

$$\|f\|_{A^p_\nu}^p \le C|f(0)|^p\widehat{\nu}(0) + C\int_0^1 M_q^p(t,f')(1-t)^{p\left(1-\frac{1}{q}\right)}\widehat{\nu}(t)dt, \quad f \in \mathcal{H}(\mathbb{D}).$$

   It allows to get rid of the integral in the expression of K_t^ω.

3. If we denote $G_\zeta^\omega(z) = \frac{d}{dz}K_\zeta^\omega(z)$, Minkowski's inequality yields

$$M_q(r,H_\omega(f)') \le \int_0^1 |f(t)|\omega(t)M_q(r,G_t^\omega)dt$$

4. Use estimates for the integral means of Bergman reproducing kernels induced by $\omega \in \widehat{\mathscr{D}}$ (Peláez-Rättyä (2016)):

$$M_q^q(r, G_t^\omega)dt \le C M_q^q(r, B_t^\omega)dt \asymp \frac{1}{\widehat{\omega}(rt)^q} \frac{1}{(1-rt)^{q-1}}$$

5. $$\|H_\omega(f)\|_{A_\nu^p}^p \le C\|f\|_{L^p_{\widehat{\nu},[0,1)}}^p + C\int_0^1 \left[\int_0^1 \frac{|f(t)|}{(1-rt)^{1-\frac{1}{q}}} \frac{\omega(t)}{\widehat{\omega}(rt)} dt\right]^p (1-r)^{p(1-\frac{1}{q})} \widehat{\nu}(r)dr$$

6. We obtain a couple of classical Hardy inequalities that hold whenever

$$\sup_{0<r<1} \left(1 + \int_0^r \frac{\widehat{\nu}(t)}{\widehat{\omega}(t)^p} dt\right)^{\frac{1}{p}} \left(\int_r^1 \left(\frac{\omega(t)}{\widehat{\nu}(t)^{\frac{1}{p}}}\right)^{p'} dt\right)^{\frac{1}{p'}} < \infty.$$

Thank you!