Extremal results, approximation properties, and related problems associated with the Loewner differential equation in $\mathbb{C}^n$

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Based on joint work with
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- **The Loewner differential equation on the unit ball in $\mathbb{C}^n$**

• I. Graham, H. Hamada, **G. K.**, M. K.: *Asymptotically spirallike mappings in several complex variables*, J. Anal. Math., **105**, 267-302 (2008).

• P. Duren, I. Graham, H. Hamada, **G. K.**: *Solutions for the generalized Loewner differential equation in several complex variables*, Math. Ann., **347**, 411-435 (2010).

- **Extremal problems for univalent mappings with parametric representation on $\mathbb{B}^n$**

• I. Graham, H. Hamada, **G. K.**, M. K. : *Extremal properties associated with univalent subordination chains in $\mathbb{C}^n$*, Math. Ann., **359**, 61–99 (2014).

Variation of Loewner chains in higher dimensions

• F. Bracci, I. Graham, H. Hamada, **G. K.**, *Variation of Loewner chains, extreme and support points in the class S^0 in higher dimensions*, Constructive Approx., **43** (2016), 231–251.

• I. Graham, H. Hamada, **G. K.**, M. K., *Support points and extreme points for mappings with A-parametric representation in $\mathbb{C}^n$*, J. Geom. Anal., **26** (2016), 1560–1595.

Approximation properties by automorphisms of $\mathbb{C}^n$ and quasiconformal diffeomorphisms in $\mathbb{C}^n$

• H. Hamada, M. Iancu, **G. K.**, S. Schleissinger, *Approximation properties of univalent mappings on the unit ball in $\mathbb{C}^n$*, J. Approx. Theory, **226** (2018), 14–33.

• H. Hamada, M. Iancu, **G. K.**, *Approximation of univalent mappings by automorphisms and quasiconformal diffeomorphisms in $\mathbb{C}^n$*, 2020.

Questions, open problems and conjectures

1. Loewner chains in several complex variables

Definition

$f : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ is an **univalent subordination chain** if:

(i) $f(\cdot, t) \equiv f_t$ is univalent (holomorphic and injective) on $\mathbb{B}^n$, $f(0, t) = 0$, $t \geq 0$.

(ii) $f(\mathbb{B}^n, s) \subseteq f(\mathbb{B}^n, t)$, $0 \leq s \leq t < \infty$.

Definition

- An univalent subordination chain $f(z, t)$ is said to be an A-normalized univalent subordination chain (A-Loewner chain) if $Df(0, t) = e^{tA}$, $t \geq 0$, where $A \in L(\mathbb{C}^n)$; $f(z, t) = e^{tA}z + \cdots$, $z \in \mathbb{B}^n$, $t \geq 0$.
- A normalized Loewner chain (Loewner chain) $f(z, t)$ is an I_n-Loewner chain. Thus, $f(z, t) = e^t z + \cdots$, $z \in \mathbb{B}^n$, $t \geq 0$.
- If, in addition , $\{e^{-t} f_t\}_{t \geq 0}$ is a *normal family* (i.e. *locally uniformly bounded family*), then $\{f_t\}_{t \geq 0}$ is called a **normal Loewner chain**.

- $H(\mathbb{B}^n) = \{f : \mathbb{B}^n \to \mathbb{C}^n \big| \, f \text{ is holomorphic } \}$.
- $S(\mathbb{B}^n) := \{h \in H(\mathbb{B}^n) : h \text{ univalent }, h(0) = 0, Dh(0) = I_n\}$.
- $S(\mathbb{B}^n)$ is closed but is not compact for $n \geq 2$.

I. Graham, H. Hamada, G. K, Canad. J. Math. (2002): The family $S^0(\mathbb{B}^n)$ of mappings in $S(\mathbb{B}^n)$ which admit parametric representation:

$$S^0(\mathbb{B}^n) := \big\{f \in S(\mathbb{B}^n) : \exists\, f(z,t) \text{ Loewner chain such that}$$

$$\{e^{-t}f(\cdot,t)\}_{t\geq 0} \text{ normal family and } f = f(\cdot,0)\big\}.$$

$$S^1(\mathbb{B}^n) := \big\{f \in S(\mathbb{B}^n) : \exists\, f(z,t) \text{ Loewner chain such that}$$

$$f = f(\cdot,0), \bigcup_{t\geq 0} f_t(\mathbb{B}^n) = \mathbb{C}^n\big\}.$$

$$S_R(\mathbb{B}^n) := \big\{f \in S(\mathbb{B}^n) : f(\mathbb{B}^n) \text{ is a Runge domain in } \mathbb{C}^n\big\}.$$

$$\mathcal{A}(\mathbb{B}^n) := \big\{\Phi \in \mathrm{Aut}(\mathbb{C}^n) : \Phi|_{\mathbb{B}^n} \in S(\mathbb{B}^n)\big\}.$$

$n = 1$

- $S(\mathbb{B}^1)$ is a compact family (i.e. closed and normal).
- $S^0(\mathbb{B}^1) = S^1(\mathbb{B}^1) = S_R(\mathbb{B}^1) = S$.
- every Loewner chain is a *normal* Loewner chain.
- for every Loewner chain[a] $\{f_t\}_{t\geq 0}$ we have: $R(f_t) = \mathbb{C}$;
- $\{f_t\}_{t\geq 0}$ is a Loewner chain if and only if $\{f_t\}_{t\geq 0}$ is a *normalized solution* of the *Loewner differential equation* associated to a *Herglotz vector field* $p : \mathbb{U} \times [0,\infty) \to \mathbb{C}$:

$$\frac{\partial f}{\partial t}(z,t) = f'(z,t)p(z,t), \quad \text{a.e. } t \geq 0,\ \forall z \in \mathbb{U}.$$

- $f \in S(\mathbb{B}^1)$ if and only if there is a Loewner chain $\{f_t\}_{t\geq 0}$: $f = f_0$.

[a] $f_t = f(\cdot, t)$.

$\mathcal{A}(\mathbb{B}^n) \subsetneq S^1(\mathbb{B}^n)$; $S^0(\mathbb{B}^n) \subsetneq S^1(\mathbb{B}^n) \subsetneq S_R(\mathbb{B}^n) = \overline{\mathcal{A}(\mathbb{B}^n)} \subsetneq S(\mathbb{B}^n)$, $n \geq 2$.

- I. Graham, H. Hamada, G.K, 2002: $S^0(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n)$, $n \geq 2$.
- I. Graham, G.K, J.A. Pfaltzgraff, 2005: $S^1(\mathbb{B}^n) \subsetneq S(\mathbb{B}^n)$, $n \geq 2$.
- L. Arosio, F. Bracci, E.F. Wold, 2013: $S^1(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n)$.
- P. Gumenyuk, 2015: $S^1(\mathbb{B}^n) \subsetneq S_R(\mathbb{B}^n)$, $n \geq 2$.
- J.E. Fornaess, E. F. Wold, 2018: $\widetilde{S}^1(\mathbb{B}^3) \subsetneq S(\mathbb{B}^3)$.

- In $\mathbb{C}^n$, $n \geq 2$, there exists $f \in S(\mathbb{B}^n)$ which cannot be embedded as the first element of a Loewner chain $f(z,t)$ such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$. (I. Graham, H. Hamada, G.K, 2002).
- In $\mathbb{C}^n$, $n \geq 2$, there exist Loewner chains $f(z,t)$ such that $\bigcup_{t\geq 0} f_t(\mathbb{B}^n) = \mathbb{C}^n$, but which are not normal, that is $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is not a normal family on $\mathbb{B}^n$. (I. Graham, H. Hamada, G.K, 2002).

- Let $f \in S(\mathbb{B}^n)$. If there is a Loewner chain $f(z,t)$ such that $f = f(\cdot, 0)$ and $\bigcup_{t\geq 0} f(\mathbb{B}^n, t) = \mathbb{C}^n$, then $f(\mathbb{B}^n)$ is a Runge domain (L. Arosio, F. Bracci, E.F. Wold, 2013).
- In $\mathbb{C}^n$, $n \geq 2$, there exist mappings $f \in S(\mathbb{B}^n)$ which cannot be embedded as the first element of a Loewner chain $f(z,t)$ with $\bigcup_{t\geq 0} f(\mathbb{B}^n, t) = \mathbb{C}^n$ (L. Arosio, F. Bracci, E.F. Wold, 2013).

- In dimension $n \geq 2$, the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \geq 0, \forall z \in \mathbb{B}^n,$$

does not have a unique normalized univalent solution $f(z,t)$ (I.Graham, G.K, J.A. Pfaltzgraff, 2005; P. Duren, I. Graham, H. Hamada, G.K, 2011)

Notations

For $A \in L(\mathbb{C}^n)$, we denote by

$$|V(A)| := \max\{|\langle A(z), z\rangle| : \|z\| = 1\},$$

$$m(A) := \min\{\mathrm{Re}\,\langle A(z), z\rangle : \|z\| = 1\},$$

$$k_+(A) := \max\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the spectrum of A.

- $k_+(A)$-the upper exponential index (Lyapunov index) of A.
- $|V(A)|$-the numerical radius of the operator A.

The Carathéodory family $\mathcal{N}_A(\mathbb{B}^n)$

$$\mathcal{N}_A := \left\{h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = A, \mathrm{Re}\,\langle h(z), z\rangle > 0, z \in \mathbb{B}^n \setminus \{0\}\right\},$$

$$\mathcal{M} := \mathcal{N}_{I_n} = \left\{h \in H(\mathbb{B}^n) : h(0) = 0, Dh(0) = I_n, \mathrm{Re}\,\langle h(z), z\rangle > 0, z \neq 0\right\}.$$

- I. Graham, H. Hamada, G. K, Canad. J. Math. (2002)

Theorem

Let $A \in L(\mathbb{C}^n)$ be s.t. $m(A) > 0$. If $f(z) = Az + \sum_{m=2}^{\infty} P_m(z) \in \mathcal{N}_A$, then
(i) $|V(P_m)| \leq 2|V(A)|$ and $\|P_m(z)\| \leq 4m|V(A)|$ for $m \geq 2$ and $\|z\| = 1$.
(ii) $m(A)r(1-r)/(1+r) \leq \|f(z)\| \leq 4|V(A)|r/(1-r)^2$ for $\|z\| = r < 1$.

Theorem

If $A \in L(\mathbb{C}^n)$ with $m(A) > 0$, then $\mathcal{N}_A$ is a compact subset of $H(\mathbb{B}^n)$.

Theorem

F. Bracci, M. Elin, D. Shoikhet, J. Nonlinear Convex Anal, 2014:
If $f \in \mathcal{N}_{I_n}$, then $\|f(z)\| \leq r\left[1 + 8\frac{r(1-r\log 2)}{(1-r)^2}\right]$, $\|z\| = r < 1$.

Open problem

Find the sharp growth result and sharp coefficient bounds for the Carathéodory family $\mathcal{N}_A(\mathbb{B}^n)$ in dimension $n \geq 2$.

- F. Bracci, CMFT (2015): **shearing process**; F. Bracci, O. Roth, 2016 I. Graham, H. Hamada, G.K, M.K, 2016 (coefficient bounds for $\mathcal{N}_A(\mathbb{B}^n)$, $m(A) > 0$);

Theorem

Let $h = (h_1, h_2) : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$h(z) = (z_1 + q^1_{0,2} z_2^2 + \cdots, z_2 + q^2_{2,0} z_1^2 + \cdots), \ z = (z_1, z_2) \in \mathbb{B}^2.$$

If $h = (h_1, h_2) \in \mathcal{M}$, then $|q^1_{0,2}| \leq \frac{3\sqrt{3}}{2}$ (sharp estimate), and $h^{[c]} \in \mathcal{M}$, where

$$h^{[c]}(z) = \left(z_1 + q^1_{0,2} z_2^2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Example

• If $f \in H(\mathbb{B}^n)$ is normalized locally univalent on $\mathbb{B}^n$, then the following conditions are equivalent:
(i) $f \in S^*(\mathbb{B}^n)$ (i.e. f is biholomorphic and $e^{-t}f(\mathbb{B}^n) \subseteq f(\mathbb{B}^n)$ for $t \geq 0$);
(ii) $f(z,t) = e^t f(z)$ is a Loewner chain on $\mathbb{B}^n \times [0,\infty)$.
(iii) $\Re\langle [Df(z)]^{-1} f(z), z\rangle > 0$, for all $z \in \mathbb{B}^n \setminus \{0\}$.

Example

• Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $\mathrm{Re}\,\langle A(z), z\rangle > 0$, $z \neq 0$. If $f \in H(\mathbb{B}^n)$ is normalized locally univalent on $\mathbb{B}^n$, then the following conditions are equivalent:
(i) f is spirallike with respect to A (i.e. f is biholomorphic on $\mathbb{B}^n$ and $e^{-tA}f(\mathbb{B}^n) \subseteq f(\mathbb{B}^n)$ for $t \geq 0$);
(ii) $f(z,t) = e^{tA} f(z)$ is A-univalent subordination chain on $\mathbb{B}^n \times [0,\infty)$.
(iii) $\Re\langle [Df(z)]^{-1} Af(z), z\rangle > 0$, for all $z \in \mathbb{B}^n \setminus \{0\}$.

• $\exists$ spirallike mapping f s.t. f is not in $S^0(\mathbb{B}^n)$. Then we introduced the family $S_A^0(\mathbb{B}^n)$.

Lemma

If $f(z,t)$ is an univalent subordination chain, then there exists a **transition mapping** $v = v(\cdot, s, t)$ associated with $f(z,t)$, such that

$$f(\cdot, s) = f(v(\cdot, s, t), t), \ 0 \leq s \leq t < \infty,$$

and $(v_{s,t})$ is **the evolution family** associated to $f(z,t)$. Moreover,

- (i) $v(\cdot, s, t)$ is univalent on $\mathbb{B}^n$ and $v(z,s,s) = z$, $z \in \mathbb{B}^n$, $s \geq 0$.
- (ii) $\|v(z,s,\cdot)\|$ is decreasing on $[s,\infty)$, for all $z \in \mathbb{B}^n$ and $s \geq 0$.
- (iii) **Semigroup property**: $v(z,s,u) = v(v(z,s,t),t,u), \quad 0 \leq s \leq t \leq u < \infty.$

• S. Reich and D. Shoikhet, Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Spaces, Imperial College Press, London, 2005.

Definition

A mapping $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is a **Herglotz vector field** if
(i) $h(\cdot,t) \in H(\mathbb{B}^n)$, $h(0,t) = 0$, and $\mathrm{Re}\,\langle h(z,t), z\rangle \geq 0$, $z \in \mathbb{B}^n$, $t \geq 0$;
(ii) $h(z,\cdot)$ is measurable on $[0,\infty)$ for $z \in \mathbb{B}^n$.

I. Graham, H. Hamada, **G.K.**, M.K., J. Anal. Math. (2008);
L. Arosio; M. Voda (2011).

Theorem

Let $A \in L(\mathbb{C}^n)$ *be such that* $m(A) > 0$*. Also let* $h : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ *satisfy the following conditions:*
(i) $h(\cdot, t) \in \mathcal{N}_A$ *for* $t \geq 0$*;*
(ii) $h(z, \cdot)$ *is measurable on* $[0, \infty)$ *for each* $z \in \mathbb{B}^n$*.*
Then for each $z \in \mathbb{B}^n$ *and* $s \geq 0$*, the initial value problem*

$$\frac{\partial v}{\partial t} = -h(v, t), \ \text{a.e. } t \geq s, \ v(z, s, s) = z, \tag{2.1}$$

has a unique solution $v = v(z, s, t)$ *such that* $v(\cdot, s, t)$ *is a univalent Schwarz mapping,* $v(z, s, \cdot)$ *is Lipschitz continuous on* $[s, \infty)$ *locally uniformly with respect to* $z \in \mathbb{B}^n$ *and* $Dv(0, s, t) = \exp(-A(t - s))$*.*

- $h(z, t)$-**Herglotz vector field** (generating vector field).

Theorem

Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. Also let $h(z,t) = Az + \cdots$ be a Herglotz vector field and let $v_{s,t}(z) = v(z,s,t)$ be the Lipschitz continuous solution on $[s,\infty)$ of (2.1). Then the limit

$$\lim_{t\to\infty} e^{tA} v_{s,t}(z) = f(z,s) \tag{2.2}$$

exists locally uniformly on $\mathbb{B}^n$ for $s \geq 0$. Moreover, $f(z,t) = e^{tA}z + \cdots$ is an A-univalent subordination chain and $\{e^{-tA}f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$ and $\bigcup_{t\geq 0} f_t(\mathbb{B}^n) = \mathbb{C}^n$. In addition, $f(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly w.r.t. $z \in \mathbb{B}^n$, and

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t),\ a.e.\ t \geq 0,\ \forall\ z \in \mathbb{B}^n. \tag{2.3}$$

- The A-univalent subordination chain $f(z,t)$ given by (2.2) is called the **canonical solution** of the Loewner differential equation (2.3).

• P. Duren, I. Graham, H. Hamada, G. Kohr, Math. Ann., 347, 411–435, 2010.

• $g(z,t)$-standard solution of the Loewner PDE (2.3) if $g(\cdot,t) \in H(\mathbb{B}^n)$, $g(0,t)=0$ and $Dg(0,t)=e^{tA}$ for $t \geq 0$, $g(z,\cdot)$ is locally Lipschitz continuous on $[0,\infty)$ locally uniformly with respect to $z \in \mathbb{B}^n$, and $g(z,t)$ is a solution of (2.3).

• When a standard solution coincides with the canonical solution?

Theorem

Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. If $f = f(z,t) : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$ is a standard solution of the Loewner PDE (2.3), i.e.,

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad a.e. \quad t \geq 0, \forall\, z \in \mathbb{B}^n,$$

then $f(\cdot,s) = f(v(\cdot,s,t),t)$, where $v(\cdot,s,t)$ is the solution of (2.1). If $\{e^{-tA}f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$, then $f(z,t)$ is an A-univalent subordination chain, and coincides with the canonical solution of (2.3).

- Every univalent solution $f(z,t) = e^{tA}z + \cdots$ of the Loewner PDE (2.3) is an A-univalent subordination chain.

Theorem

Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. If $f(z,t) = e^{tA}z + \cdots$, then $f(z,t)$ is an A-univalent subordination chain if and only if $f(z,t)$ is a univalent solution of the Loewner differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad a.e. \quad t \geq 0, \forall\, z \in \mathbb{B}^n,$$

associated with a Herglotz vector field $h : \mathbb{B}^n \times [0,\infty) \to \mathbb{C}^n$.

- The uniqueness result about the canonical solution and the relation (2.2) do not hold if $k_+(A) = 2m(A)$ (Duren, Graham, Hamada, G.K, 2010).

Theorem

If $f(z,t) = e^t z + \cdots$ is a Loewner chain such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$, then $\bigcup_{t\geq 0} f_t(\mathbb{B}^n) = \mathbb{C}^n$ and

$$\frac{e^t\|z\|}{(1+\|z\|)^2} \leq \|f(z,t)\| \leq \frac{e^t\|z\|}{(1-\|z\|)^2}, \ z \in \mathbb{B}^n, \ t \geq 0.$$

- There exist Loewner chains $f(z,t)$ which do not satisfy this growth result and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is not a normal family on $\mathbb{B}^n$, for $n \geq 2$.

Example

Let $g(z,t) = \left(\frac{e^t z_1}{(1-z_1)^2}, \frac{e^t z_2}{(1-z_2)^2}\right)$ for $z = (z_1, z_2) \in \mathbb{B}^2$, $t \geq 0$. Then $g(z,t)$ is a Loewner chain, and if $\Phi(z) = (z_1, z_2 + z_1^2)$, then $\Phi \in \mathrm{Aut}(\mathbb{C}^2)$ and $f(z,t) := \Phi(g(z,t))$ is a Loewner chain such that $\|f(r,0)\| > r/(1-r)^2$ for $r \in (0,1)$, and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is not a normal family on $\mathbb{B}^2$.

- $f(z,t)$ and $g(z,t) \mapsto$ the same Loewner PDE
- $\bigcup_{t\geq 0} f_t(\mathbb{B}^2) = \bigcup_{t\geq 0} g_t(\mathbb{B}^2) = \mathbb{C}^2$.

- P. Duren, I. Graham, H. Hamada and G.K, Math. Ann. (2010)

Theorem

Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. Let $h(z,t) = Az + \cdots$ be a Herglotz vector field, and let $f(z,t) = e^{tA}z + \cdots$ be the canonical solution of the Loewner PDE (2.3), i.e.

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad \text{a.e. } t \geq 0, \forall z \in \mathbb{B}^n,$$

and let $g(z,t)$ be a standard solution of (2.3). If $\{e^{-tA}g(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$, then there exists $\Psi \in L(\mathbb{C}^n)$ such that $g(z,t) = \Psi(f(z,t))$ for $z \in \mathbb{B}^n$ and $t \geq 0$.

Corollary

Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. The canonical solution $f(z,t) = e^{tA}z + \cdots$ of the Loewner PDE (2.3) is the unique normalized univalent Loewner chain solution such that $\{e^{-tA}f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$, and $R(f_t) := \bigcup_{t\geq 0} f(\mathbb{B}^n,t) = \mathbb{C}^n$.

- How does look a standard solution in terms of the canonical solution?
  - P. Duren, I. Graham, H. Hamada and G.K., Math. Ann. (2010)

Theorem

Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. If $g(z,t) = e^{tA}z + \cdots$ is a Loewner chain, then there exist a unique Loewner chain $f(z,t) = e^{tA}z + \cdots$ such that $\{e^{-tA}f(\cdot,t)\}_{t\geq 0}$ is a normal family, and a unique entire univalent mapping Φ (Φ is an automorphism of $\mathbb{C}^n$ or a Fatou-Bieberbach map) such that $g(\cdot,t) = \Phi(f(\cdot,t))$, $t \geq 0$.

- L. Arosio (2011); M. Vodă (2011) (resonances).
- L. Arosio, F. Bracci, E.F. Wold, *Solving the Loewner PDE in complete hyperbolic starlike domains of* $\mathbb{C}^n$. Adv. Math. 242, 209–216 (2013).

• P. Duren, I. Graham, H. Hamada, G. Kohr: *Solutions for the generalized Loewner differential equation in several complex variables*, Math. Ann., 347 (2010).

1. Every **normal Loewner chain**[a] $(g_t)_{t\geq 0}$ satisfies: $R(g_t) = \mathbb{C}^n$.
2. For every **Loewner chain** $(g_t)_{t\geq 0}$ there exists a unique **normal Loewner chain** $\{f_t\}_{t\geq 0}$ and a unique biholomorphic mapping $\Phi : \mathbb{C}^n \to R(f_t)$ such that $\boxed{g_t = \Phi \circ f_t}$, for all $t \geq 0$.
   If $R(g_t) = \mathbb{C}^n$, then Φ is an *automorphism* of $\mathbb{C}^n$. If $R(g_t) \neq \mathbb{C}^n$, then Φ is a *Fatou-Bieberbach mapping*.
3. $(g_t)_{t\geq 0}$ is **Loewner chain** if and only if $(g_t)_{t\geq 0}$ is a *normalized solution* of the **Loewner differential equation** associated to a *Herglotz vector field* $h : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$:

$$\frac{\partial g}{\partial t}(z,t) = Dg(z,t)h(z,t),, \quad \text{a.e. } t \geq 0, \forall z \in \mathbb{B}^n.$$

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[a] $g_t := g(\cdot, t)$.

2. Parametric representation on the unit ball

Definition

Let $f \in H(\mathbb{B}^n)$ be such that $f(0) = 0$, $Df(0) = I_n$. Also let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. We say that f has A-parametric representation if there exists mapping $h : \mathbb{B}^n \times [0, \infty) \to \mathbb{C}^n$ such that

(i) $h(\cdot, t) \in \mathcal{N}_A(\mathbb{B}^n)$, $t \geq 0$;

(ii) $h(z, \cdot)$ is measurable on $[0, \infty)$, $z \in \mathbb{B}^n$

such that $f(z) = \lim_{t\to\infty} e^{tA} v(z, t)$ locally uniformly on $\mathbb{B}^n$, where $v = v(z, t)$ is the unique Lipschitz continuous solution on $[0, \infty)$ of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v, t), \text{ a.e. } t \geq 0, \ v(z, 0) = z, \forall z \in \mathbb{B}^n.$$

- $S^0_A(\mathbb{B}^n) := \{f \in S(\mathbb{B}^n) : f \text{ has A-parametric representation}\}$
- $S^0(\mathbb{B}^n) := S^0_{I_n}(\mathbb{B}^n)$ (usual parametric representation).

I. Graham, H. Hamada, G.K., M.K., 2008:

Theorem

Let $f \in S(\mathbb{B}^n)$ and $A \in L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$. Then $f \in S_A^0(\mathbb{B}^n)$ if and only if there exists an A-univalent subordination chain $f(z,t)$ such that $\{e^{-tA}f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$ and $f = f(\cdot,0)$.

Corollary

Let $f \in S^0(\mathbb{B}^n) = S^0_{I_n}(\mathbb{B}^n)$. *Then*

$$\frac{\|z\|}{(1+\|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1-\|z\|)^2},\ z \in \mathbb{B}^n.$$

These estimates are sharp.

Theorem

$S_A^0(\mathbb{B}^n)$ is a compact family for $A \in L(\mathbb{C}^n)$ with $k_+(A) < 2m(A)$.

- This compactness result does not hold if $k_+(A) = 2m(A)$ (Duren et al. 2010).

3. Rungeness and approximation properties

H. Hamada, M. Iancu, G.K, S. Schleissinger (2017-2018).

$\mathrm{Aut}(\mathbb{C}^n) = \{\Phi : \mathbb{C}^n \to \mathbb{C}^n : \Phi$ is an automorphism of $\mathbb{C}^n\}$,

$\mathcal{A}(\mathbb{B}^n) = \{\Phi|_{\mathbb{B}^n} : \Phi \in \mathrm{Aut}(\mathbb{C}^n), \Phi(0) = 0, D\Phi(0) = I_n\}$.

Definition

Let $D_1 \subseteq D_2 \subseteq \mathbb{C}^n$ be domains. Then (D_1, D_2) is called a Runge pair if $\mathcal{O}(D_2)$ is dense in $\mathcal{O}(D_1)$, where $\mathcal{O}(D_j)$ is the family of holomorphic functions of D_j into $\mathbb{C}$, for $j = 1, 2$. A domain $D \subseteq \mathbb{C}^n$ is called Runge if $(D, \mathbb{C}^n)$ is a Runge pair.

Remark

(i) A domain $D \subseteq \mathbb{C}$ is Runge if and only if D is simply connected.
(ii) Every starlike domain $D \subseteq \mathbb{C}^n$ is Runge (A. El Kasimi, Complex Var. Theory Appl., 1988).
(iii) Every spirallike domain $D \subseteq \mathbb{C}^n$ with respect to $A \in L(\mathbb{C}^n)$ with $m(A) > 0$ is Runge (H. Hamada, Adv. Math., 2015).

• J. Wermer, 1959: There is a biholomorphic mapping $F : \mathbb{U}^2 \to \mathbb{C}^2$ such that $F(\mathbb{U}^2)$ is not Runge, and thus the Runge property is not an invariant property with respect to biholomorphic maps.

• E. Fornaess Wold, 2008: For $n \geq 2$, there is a Fatou-Bieberbach mapping $\psi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\psi(\mathbb{C}^n)$ is not Runge.

• Every automorphism of $\mathbb{C}^n$ preserves the Runge property.

• E. Andersén, L. Lempert, *On the group of holomorphic automorphisms of* $\mathbb{C}^n$, Invent. Math., **110** (1992), 371–388.
• F. Forstnerič, J.P. Rosay, *Approximation of biholomorphic mappings by automorphisms of* $\mathbb{C}^n$, Invent. Math. 112 (1993) 323–349.

Theorem

Let $n \geq 2$, $D \subseteq \mathbb{C}^n$ be a starlike domain w.r.t. the origin, and let $f : D \to \mathbb{C}^n$ be a biholomorphic mapping. Then $f(D)$ is a Runge domain if and only if there is a sequence of automorphisms of $\mathbb{C}^n$ which converges to f, locally uniformly on D.

• H. Hamada, *Approximation properties on spirallike domains of* $\mathbb{C}^n$, Adv. Math., **268** (2015), 467–477.

Theorem

Let $n \geq 2$. Let $D \subseteq \mathbb{C}^n$ be a spirallike domain with respect to A with $k_+(A) < 2m(A)$, and let $f : D \to \mathbb{C}^n$ be a biholomorphic mapping. Then $f(D)$ is a Runge domain if and only if there is a sequence of automorphisms of $\mathbb{C}^n$ which converges to f, locally uniformly on D.

• L. Arosio, F. Bracci, E. Fornaess Wold, Adv. Math., 2013:

Theorem

Every Loewner chain $(f_t)_{t\geq 0}$ satisfies: $(f_s(\mathbb{B}^n), f_t(\mathbb{B}^n))$ is a Runge pair, $0 \leq s \leq t$. In particular, if $R(f_t) := \bigcup_{t\geq 0} f_t(\mathbb{B}^n) = \mathbb{C}^n$, then $f_t(\mathbb{B}^n)$ is a Runge domain in $\mathbb{C}^n$, for all $t \geq 0$.

• $S^0(\mathbb{B}^n) \subsetneq S^1(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n)$, $n \geq 2$.

$S_R(\mathbb{B}^n) = \{f \in S(\mathbb{B}^n) : f(\mathbb{B}^n) \text{ is Runge}\}.$

$$S_A^0(\mathbb{B}^n) = \{f \in S(\mathbb{B}^n) : \exists\ A\text{-normal Loewner chain } (f_t),\ f = f_0\}.$$

$$S_A^1(\mathbb{B}^n) = \{f \in S(\mathbb{B}^n) : \exists\ A\text{-Loewner chain } (f_t) \text{ with } R(f_t) = \mathbb{C}^n \text{ such that } f(\cdot, 0) = f\}$$

$$\mathcal{A}(\mathbb{B}^n) = \{\Phi|_{\mathbb{B}^n} : \Phi \in \mathrm{Aut}(\mathbb{C}^n), \Phi(0) = 0, D\Phi(0) = I_n\}.$$

- For $n = 1$: $S^0 = S^1 = S_R = S$ and $\mathcal{A} = \{id\}$.
- For $n \geq 2$: $S^0(\mathbb{B}^n) \subsetneq S^1(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n)$.
- If $f \in S_R(\mathbb{B}^n) \cap C^1(\overline{\mathbb{B}^n})$, then $f \in S^1(\mathbb{B}^n)$ (an application of the variational method due to Bracci, Graham, Hamada, G.K., 2016).

- L. Arosio, F. Bracci, E.F. Wold, Proc. Amer. Math. Soc. (2015): Let $n \geq 2$ and $A \in L(\mathbb{C}^n)$, $k_+(A) < 2m(A)$. Then

$$\mathcal{A}(\mathbb{B}^n) \subsetneq S_A^1(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n) = \overline{\mathcal{A}(\mathbb{B}^n)} \subsetneq S(\mathbb{B}^n).$$

- S. Schleissinger (2014); H.Hamada (2015): $S_A^0(\mathbb{B}^n) \subseteq S_R(\mathbb{B}^n)$.

• H. Hamada, Adv. Math., 2015: A-spirallike domains in $\mathbb{C}^n$ are Runge domains, for all $A \in L(\mathbb{C}^n)$, $m(A) > 0$.

H. Hamada, M. Iancu, G. K, S. Schleissinger (2018):

Theorem

Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$, and let $n \geq 2$. Then $S_A^0(\mathbb{B}^n) = \overline{S_A^0(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}$. In particular, $S^0(\mathbb{B}^n) = \overline{S^0(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}$.

$$S^*(\mathbb{B}^n) = \overline{S^*(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}.$$

Theorem

If $n \geq 2$, then $K(\mathbb{B}^n) = \overline{K(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}$.

H. Hamada, M. Iancu, G.K (2018):

- If $n \geq 2$ and $m(A) > 0$, then $\overline{S_A^0(\mathbb{B}^n)} = \overline{S_A^0(\mathbb{B}^n) \cap \mathcal{A}(\mathbb{B}^n)}$.

4. Extreme points, support points and parametric representation

Definition

Let X be a locally convex topological vector space and let $E \subseteq X$.
(i) A point $x \in E$ is called an *extreme point* of E ($x \in \mathrm{ex}\, E$) provided $x = ty + (1-t)z$, where $t \in (0,1)$, $y, z \in E$, implies $x = y = z$. That is, $x \in E$ is an extreme point of E if x is not a proper convex combination of two points in E.
(ii) A point $w \in E$ is called a *support point* of E ($w \in \mathrm{supp}\, E$) if $\Re L(w) = \max_{y \in E} \Re L(y)$ for some continuous linear functional $L : X \to \mathbb{C}$ such that $\Re L$ is nonconstant on E.

- $X := H(\mathbb{B}^n)$; $\mathrm{ex}\, S_A^0(\mathbb{B}^n) \neq \emptyset$ and $\mathrm{supp}\, S_A^0(\mathbb{B}^n) \neq \emptyset$ for $k_+(A) < 2m(A)$.
- J. Muir, T.J. Suffridge (2006): extreme points for $K(\mathbb{B}^n)$.
- I. Graham, G.K and J.A, Pfaltzgraff (2005): extreme and support points of compact subsets of $S(\mathbb{B}^n)$ generated by extension operators.

I. Graham, H. Hamada, G.K, M.K (2010-2014):

Theorem 1

Let $f \in \mathrm{ex}\, S^0(\mathbb{B}^n)$. Also, let $f(z,t)$ be a Loewner chain such that $f = f(\cdot,0)$ and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$. Then $e^{-t}f(\cdot,t) \in \mathrm{ex}\, S^0(\mathbb{B}^n)$ for $t \geq 0$.

Theorem 2

Let $f \in \mathrm{supp}\, S^0(\mathbb{B}^n)$ and $f(z,t)$ be a Loewner chain s.t. $f = f(\cdot,0)$ and $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is a normal family on $\mathbb{B}^n$. Then $\exists\ t_0 > 0$ s.t. $e^{-t}f(\cdot,t) \in \mathrm{supp}\, S^0(\mathbb{B}^n)$ for $0 \leq t < t_0$.

• If no bounded map in $S^0_A(\mathbb{B}^n)$ is a support point of $S^0_A(\mathbb{B}^n)$, then Theorem 2 holds for all $t \in [0,\infty)$.

Conjecture 1 (2010)

There are no bounded support/extreme points of $S^0(\mathbb{B}^n)$ for $n \geq 2$.

- S. Schleissinger, Proc. AMS, 2014 (Runge pairs in $\mathbb{C}^n$):

Theorem

If $f \in \operatorname{supp} S^0(\mathbb{B}^n)$ then $e^{-t}f(\cdot, t) \in \operatorname{supp} S^0(\mathbb{B}^n)$ for $t \geq 0$.

- F. Bracci, CMFT, 2015: in higher dimensions, there exist bounded support points for the family $S^0(\mathbb{B}^n)$.

Theorem

If $f = (f_1, f_2) \in S^0(\mathbb{B}^2)$, then $|a^1_{0,2}| \leq \frac{3\sqrt{3}}{2}$, where $a^1_{0,2} = \frac{\partial^2 f_1}{\partial z_2^2}(0)$. The above bound is sharp and equality holds for

$$F(z) = \left(z_1 + \frac{3\sqrt{3}}{2} z_2^2, z_2\right), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

In particular, F is a bounded support point of $S^0(\mathbb{B}^2)$.

- I. Graham, H. Hamada, **G.K.**, M.K., Math. Ann., 2014:

Theorem

Let $A \in L(\mathbb{C}^n)$ be such that $k_+(A) < 2m(A)$. Also let $f \in \text{supp}\, S_A^0(\mathbb{B}^n)$ (resp. $f \in \text{ex}\, S^0(\mathbb{B}^n)$) and let $f(z,t)$ be an A-normalized univalent subordination chain such that $f = f(\cdot, 0)$ and $\{e^{-tA} f(\cdot, t)\}_{t \geq 0}$ is a normal family on $\mathbb{B}^n$. Then $e^{-tA} f(\cdot, t) \in \text{supp}\, S_A^0(\mathbb{B}^n)$ (respectively $e^{-tA} f(\cdot, t) \in \text{ex}\, S_A^0(\mathbb{B}^n)$) for $t \geq 0$.

- Extension to higher dimensions of results due to R. Pell, 1980; W.E. Kirwan, 1980, and Kirwan and Schober, 1982, for $n = 1$ (the family S).
- Extension of Schleissinger's result, 2014.

Open problems

- Is it true that $S(\mathbb{B}^n) = \tilde{S}^1(\mathbb{B}^n)$ for $n \geq 2$?
- Characterize $\mathrm{ex}\mathcal{N}_A$ and $\mathrm{supp}\,\mathcal{N}_A$ in dimension $n \geq 2$, where $k_+(A) < 2m(A)$. Connections with $\mathrm{ex}\, S_A^0(\mathbb{B}^n)$ and $\mathrm{supp}\, S_A^0(\mathbb{B}^n)$.
- Find connections between $\mathrm{ex}\, S^0(\mathbb{B}^n)$ (resp. $\mathrm{supp}\, S^0(\mathbb{B}^n)$) and $\mathrm{ex}\,\mathcal{M}$ (resp. $\mathrm{supp}\,\mathcal{M}$).
- If $f \in \mathrm{ex}\, S^0(\mathbb{B}^n)$, is it true that f is unbounded on $\mathbb{B}^n$, $n \geq 2$?
- Characterize the extreme/support points of other compact subsets of $S^0(\mathbb{B}^n)$ (for example, the families of convex and starlike mappings).

Question

Let $f \in S^0(\mathbb{B}^n)$ and let (f_t) be a Loewner chain such that $f = f_0$. Also, let $h = h(z, t)$ be the associated Herglotz vector field of (f_t).

(i) Is it true that $h(\cdot, t) \in \mathrm{ex}\,\mathcal{M}(\mathbb{B}^n)$ for a.e. $t \in [0, \infty)$ if and only if $f \in \mathrm{ex}\, S^0(\mathbb{B}^n)$?

(ii) If $h(\cdot, t) \in \mathrm{supp}\,\mathcal{M}(\mathbb{B}^n)$ for a.e. $t \geq 0$, is it true that $f \in \mathrm{supp}\, S^0(\mathbb{B}^n)$?

- O. Roth, 2016: If $f \in \mathrm{supp}\, S^0(\mathbb{B}^n)$, then $h(\cdot, t) \in \mathrm{supp}\,\mathcal{M}(\mathbb{B}^n)$ for a.e. $t \geq 0$.

- Characterize all bounded support points of $S^0(\mathbb{B}^n)$, for $n \geq 2$.

Thank you for your attention!