Approximation and closed ideals in some analytic spaces

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- **The Hardy space** H^2 is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathrm{H}^2}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \log(1/|z|) dA(z) < \infty.$$

Problem

Let $X \subset \mathrm{H}^2$ be a Banach algebra which has the division property.

Describe the closed ideals of the algebra X.

Notations

- Let $\mathcal{I}$ be a non trivial closed ideal of X. We denote by
  - $\theta_{\mathcal{I}}$ the greatest common divisor of the inner parts of nonzero functions in $\mathcal{I}$.
  - $E_{\mathcal{I}} := \{\zeta \in \mathbb{T} : f(\zeta) = 0, \text{ for all } f \in \mathcal{I}\}$.

- $X = A(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$ equipped by the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

is a commutative algebra.

Theorem ([1], [2])

Let $\mathcal{I}$ be a non trivial closed ideal of $A(\mathbb{D})$. Then

$$\mathcal{I} = \left\{ f \in A(\mathbb{D}) : \ f_{|E_\mathcal{I}} = 0 \text{ and } f \in \theta_\mathcal{I} A(\mathbb{D}) \right\}.$$

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1. A. Beurling,
On two problems concerning linear transformations in Hilbert space
Acta Mathematica, **(1) :** 239–255, 1949.

2. W. Rudin,
The closed ideals in an algebra of analytic functions.
Canadian Journal of Mathematics, **(9) :** 426–434, 1957.

- $X = \lambda_\alpha$, for $\alpha \in (0,1)$. The separable Lipschitz analytic algebra λ_α is given by
$$\lambda_\alpha := \{f \in A(\mathbb{D}) : \ |f(z) - f(w)| = o(|z-w|^\alpha)\}.$$

Equipped with the norm,

$$\|f\|_{\lambda_\alpha} = \|f\|_{\mathrm{H}^\infty} + \sup_{z,w\in\mathbb{D}} \frac{|f(z) - f(w)|}{|z-w|^\alpha},$$

λ_α is a Banach algebra.

Matheson([3])

Let $\mathcal{I}$ be a non trivial closed ideal of λ_α. Then

$$\mathcal{I} = \left\{f \in \lambda_\alpha : \ f_{|E_{\mathcal{I}}} = 0, \text{ and } f \in \theta_{\mathcal{I}}\lambda_\alpha\right\}.$$

3. AL. Matheson, Closed ideals in rings of analytic functions satisfying a Lipschitz condition. *Banach spaces of analytic functions*, **67–72**, 1977.

- $X = \mathcal{D} \cap \lambda_\alpha$

$$\mathcal{D} \cap \lambda_\alpha := \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{\mathcal{D} \cap \lambda_\alpha} = \|f\|_{\lambda_\alpha} + \mathcal{D}^{1/2}(f) < \infty\}.$$

where

$$\mathcal{D}(f) = \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 dA(z), \quad \text{and} \quad \|f\|_{\lambda_\alpha} := \|f\|_\infty + \sup_{z \in \mathbb{D}} \frac{|f'(z)|}{(1-|z|)^{1-\alpha}}$$

The space $\mathcal{D} \cap \lambda_\alpha$ equipped with the norm $\|.\|_{\mathcal{D} \cap \lambda_\alpha}$ is a Banach algebra.

Theorem ([4])

Let $\mathcal{I}$ be a non trivial closed ideal of $\mathcal{D} \cap \lambda_\alpha$. Then

$$\mathcal{I} = \left\{ f \in \mathcal{D} \cap \lambda_\alpha \ : \ f_{|E_{\mathcal{I}}} = 0, \text{ and } f \in \theta_{\mathcal{I}} \mathcal{D} \cap \lambda_\alpha \right\}.$$

4. B. Bouya, Closed Ideals in Some Algebras of Analytic Functions *Canad. J. Math*, **(61) :** 282–298, 2009.

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Let $\alpha \in (0, 1)$.

- M. Matheson : $X = \lambda_\alpha$, Michigan Math. J. (1978).
- B. Bouya : $X = \mathcal{D} \cap \lambda_\alpha$, Canad. J. Math.(2009).

Theorem

Let $f \in X$ be a function that vanishes on a closed subset E of $\mathbb{T}$. Then, given $M > 0$, there exists $(f_n)_n \subset X$ such that

1. f_n vanishes on E for all $n \geq 1$.
2. $|f_n(z)| = O(dist^M(z, E))$ for all $n \geq 1$.
3. $\lim\limits_{n \to 0} \|f_n f - f\|_X = 0$.

- $X = \lambda_\alpha$.

By arguing in the same way as in ([5]), one can see that

$$\|f\|_\alpha \asymp \|f\|_\infty + \sup_{\zeta_1,\zeta_2\in\mathbb{T},\zeta_1\neq\zeta_2} \frac{||f(\zeta_1)|-|f(\zeta_2)||}{|\zeta_1-\zeta_2|^\alpha} + \sup_{z\in\mathbb{D}} \frac{P(|f|)(z)-|f(z)|}{(1-|z|)^\alpha}.$$

Let $f,\ g$ be two outer functions. Let $f\wedge g$ be the outer function associated to $|f|\wedge|g|(\zeta) := \min(|f(\zeta)|,|g(\zeta)|), (\zeta\in\mathbb{T})$. Namely,

$$f\wedge g(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\log(|f|\wedge|g|(\zeta))dm(\zeta)\right), \quad (z\in\mathbb{D}).$$

Theorem [H.B.I & O. El-Fallah]

Let $p\geq 1$. Let $f\in H^2$ be an outer function. We have

$$P(|f\wedge f^p|)(z) - |f\wedge f^p(z)| \leq p^2\left(P(|f|)(z)-|f(z)|\right), \quad (z\in\mathbb{D}).$$

Let $x\geq 1$ and $y\in[0,1]$. Let $a,b\geq 0$ be such that $a+b=1$. Then

$$x^a(p^2y^b - y^{pb}) \leq b(p^2y - y^p) + a(p^2-1)x.$$

5. K.M. Dyakonov, Equivalent norm on Lipschitz-type spaces of homorphic functions *Acta mathematica, 1997.*

Let $f = \theta g \in \lambda_\alpha \setminus \{0\}$, where θ, g are respectively the inner and outer factors of f. Suppose that $f_{|E} = 0$. Since λ_α possesses the division property, $g \in \lambda_\alpha$. We have

$$g_{|E} = 0 \quad and \quad |g(z)| = O(\mathrm{dist}(z, E)^\alpha).$$

Let $n \geq 1$, $p = 2^k$, $k := E(\ln(1 + M/\alpha)/\ln(2)) + 1$, and put $f_n = 1 \wedge n^{p-1} g^{p-1}$. So

$$f_n \in \lambda_\alpha \quad and \quad |f_n(z)| = O(\mathrm{dist}(z, E)^M).$$

We have $ff_n = \theta(g \wedge n^{p-1} g^p)$, $(ff_n)_n$ converges uniformly to f on any compact subset of $\mathbb{D}$. We claim that

$$||ff_n(\zeta_1)| - |ff_n(\zeta_2)|| \leq p\,||f(\zeta_1)| - |f(\zeta_2)||\,, \qquad (\zeta_1, \zeta_2 \in \mathbb{T}).$$

$$P(|ff_n|)(z) - |ff_n(z)| \leq p^2\,(P(|f|)(z) - |f(z)|)\,, \qquad (z \in \mathbb{D}).$$

This means that

1. $(ff_n)_n$ converges to f uniformly on compact subsets of $\mathbb{D}$,
2. $||ff_n(\zeta_1)| - |ff_n(\zeta_2)|| = o(|\zeta_1 - \zeta_2|^\alpha)$, $\quad |\zeta_1 - \zeta_2| \to 0$, $\quad \zeta_1, \zeta_2 \in \mathbb{T}$, uniformly in n,
3. $P(|ff_n|)(z) - |ff_n(z)| = o((1 - |z|)^\alpha)$, $|z| \to 1^-$, uniformly in n.

Therefore $(ff_n)_n$ converges to f in λ_α.

The standard Dirichlet space $\mathcal{D}_\gamma$, $\gamma \in (0,1)$, is defined by

$$\mathcal{D}_\gamma := \{f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{D}_\gamma(f) := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^\gamma dA(z) < \infty\}.$$

Let f be a function in H^2. By Green's formula, we have

$$\mathcal{D}_\gamma(f) = \int_{\mathbb{D}} \left(P(|f|^2)(z) - |f(z)|^2\right) d\mu_\gamma(z),$$

where $d\mu_\gamma(z) := \frac{1}{4}|\Delta(1-|z|^2)^\gamma| dA(z)$ and

$$P(|f|^2)(z) = \int_{\mathbb{T}} |f^*(\zeta)|^2 \frac{1-|z|^2}{|\zeta - z|^2} dm(\zeta).$$

The space $\mathcal{D}_\gamma \cap \lambda_\alpha$ equipped with the norm

$$\|f\|_{\mathcal{D}_\gamma \cap \lambda_\alpha} = \|f\|_{\lambda_\alpha} + \mathcal{D}_\gamma^{1/2}(f)$$

is a Banach algebra.

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- $X = \mathcal{D}_\gamma \cap \lambda_\alpha$, where $\gamma, \alpha \in (0,1)$.

Theorem ([6])

Let $f \in \mathcal{D}_\gamma \cap \lambda_\alpha$ which vanishes on a closed subset E. Then, given $M > 0$, there exists $(f_n)_n \subset \mathcal{D}_\gamma \cap \lambda_\alpha$ such that

1. f_n vanishes on E.
2. $|f_n(z)| = O(dist^M(z,E))$ for all $n \geq 1$.
3. $\lim_{n\to 0} \|f_n f - f\|_{\mathcal{D}_\gamma \cap \lambda_\alpha} = 0$.

- M. Matheson : $X = \lambda_\alpha$, Michigan Math. J. (1978).
- B. Bouya : $X = \mathcal{D} \cap \lambda_\alpha$, Canad. J. Math.(2009).

6. H. Bahajji-El Idrissi and O. El-Fallah, Approximation in spaces of analytic functions. *Studia Mathematica, 2019.*

Let $f = \theta g \in \mathcal{D}_\gamma \cap \lambda_\alpha \setminus \{0\}$, where θ, g are respectively the inner and outer factors of f. Suppose that $f_{|E} = 0$. Since λ_α possesses the F-property, $g \in \lambda_\alpha$. We have $g_{|E} = 0$
Let $n \geq 1$ and $p = 2^k$ where $k := E(\frac{\ln(1+M/\alpha)}{\ln(2)}) + 1$. We consider

$$ff_n = \theta(g \wedge n^{p-1} g^p)$$

From the above discussion $f_n \in \mathcal{D}_\gamma \cap \lambda_\alpha$ and clearly

$$|f_n(z)| = O(\text{dist}(z, E)^M).$$

As we said before $(ff_n)_n$ converges to f in λ_α. Next, from the above discussion, we get

$$\begin{aligned} \mathcal{D}_\gamma(ff_n) &= \int_{\mathbb{D}} (P(|ff_n|^2)(z) - |ff_n(z)|^2) d\mu_\gamma(z) \\ &\leq p^2 \int_{\mathbb{D}} \left(P(|f|^2)(z) - |f(z)|^2\right) d\mu_\gamma(z) = p^2 \mathcal{D}_\gamma(f). \end{aligned}$$

Then $(ff_n)_n$ converges weakly to $f \in \mathcal{D}_\gamma$. Let F_j be the convex hull of $\{ff_n : \ n \geq j\}$. Since $\mathcal{D}_\gamma$ is a Hilbert space there exists $fg_j \in F_j$ such that $\lim_j \mathcal{D}_\gamma(fg_j - f) = 0$. Since $(ff_n)_n$ converges to f in λ_α, we also have $\lim_j \|fg_j - f\|_\alpha = 0$.

- $X = \mathcal{D} \cap \lambda_\alpha$

Theorem

Let $f = B.S_\sigma.g$ be a function in H^2. Then

1. S. Richter & C. Sundberg. :

$$\mathcal{D}(g \wedge g^p) \leq 2^p \mathcal{D}(g), \qquad (p = 2^k).$$

2. L. Carleson :

$$\mathcal{D}(f) = \mathcal{D}(g) + \sum_n \int_{\mathbb{T}} \frac{1-|z_n|^2}{|z_n-\zeta|^2} |g(\zeta)|^2 dm(\zeta) + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|g(\zeta)|^2}{|\lambda-\zeta|^2} d\sigma(\lambda) dm(\zeta),$$

Let (ff_n) the sequence defined above, $ff_n = \theta(g \wedge n^{p-1} g^p)$, we have

$$\begin{aligned} \mathcal{D}(ff_n) &\leq p^2 \mathcal{D}(g) + \sum_n \int_{\mathbb{T}} \frac{1-|z_n|^2}{|z_n-\zeta|^2} |g(\zeta)|^2 dm(\zeta) + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|g(\zeta)|^2}{|\lambda-\zeta|^2} d\sigma(\lambda) dm(\zeta) \\ &\leq p^2 \mathcal{D}(f). \end{aligned}$$

1. M_z is a bounded operator on $\mathcal{D}_\gamma$ and it checks the following inequality

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \|M_z^k f\|^2_{\mathcal{D}_\gamma} \leq 0, \qquad \text{for all } f \in \mathcal{D}_\gamma \text{ and for all } n \geq 2.$$

2. M_z is an operator analytic and cyclic on $\mathcal{D}_\gamma$, namely
   - (*i*) $\bigcap_n M_z^n \mathcal{D}_\gamma = \{0\}$,
   - (*ii*) $\dim(\mathcal{D}_\gamma \ominus M_z(\mathcal{D}_\gamma)) = 1$.

Theorem ([7])

Let T be a bounded operator analytic and cyclic on a separated Hilbert space $\mathcal{H}$ such that

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \|T^k x\|^2_{\mathcal{H}} \leq 0 \qquad \forall x \in \mathcal{H} \text{ and } \forall n \geq 2.$$

Then there exists a positive superharmonic function ω on $\mathbb{D}$ such that $(T, \mathcal{H})$ is unitarily equivalent to $(M_z, \mathcal{D}_\omega)$.

- S. Richter, for $n = 2$ and if we replace the inequality with an equality.

7.

A. Aleman,
The multiplication operator on Hilbert spaces of analytic functions.
Habilitation, Uppsala., **1993.**

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Let ω be a positive superharmonic weight on $\mathbb{D}$. **The superharmonically weighted Dirichlet space** $\mathcal{D}_\omega$ associated with ω is defined by

$$\mathcal{D}_\omega := \left\{ f \in \mathrm{H}^2 : \mathcal{D}_\omega(f) := \int_{\mathbb{D}} |f'(z)|^2 \omega(z) dA(z) < \infty \right\}.$$

The space $\mathcal{D}_\omega$ will be endowed by the hilbertian norm $\|f\|_\omega^2 := \|f\|_{H^2}^2 + \mathcal{D}_\omega(f)$.
Let ω be a nonnegative superharmonic weight on $\mathbb{D}$. By Jensen-Riesz representation theorem there exists a positive Borel measure μ on $\mathbb{D}$ and a finite positive Borel measure ν on $\mathbb{T}$ such that

$$\omega(z) = U_\mu(z) + P_\nu(z), \quad (z \in \mathbb{D}),$$

where U_μ is the Green potential of μ and P_ν is the Poisson transform of ν, namely

$$U_\mu(z) = \int_{\mathbb{D}} \log \left| \frac{1 - \overline{w}z}{w - z} \right| d\mu(w) \quad \text{and} \quad P_\nu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|1 - \overline{\zeta}z|^2} d\nu(\zeta).$$

It is well known that $U_\mu \neq \infty$ if and only if $\int_{\mathbb{D}} (1 - |z|) d\mu(z) < \infty$.

$$\int_{\mathbb{D}} (1 - |z|) d\mu(z) < \infty.$$

It is clear, from Green's formula, that

$$\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right| dA(w) = P(|f|^2)(z) - |f(z)|^2 \quad (z \in \mathbb{D}).$$

So, we have the following formula

$$\mathcal{D}_{U_\mu}(f) = \int_{\mathbb{D}}\int_{\mathbb{D}} |f'(w)|^2 \log\left|\frac{1-z\overline{w}}{z-w}\right| dA(w)d\mu(z) = \int_{\mathbb{D}} \left(P(|f|^2)(z) - |f(z)|^2\right) d\mu(z)$$

Theorem

Let f be an outer functions in H^2 . Then

1. H.B.I. & O. El-Fallah,

$$\mathcal{D}_{U_\mu}(f \wedge f^p) \leq p^2 \mathcal{D}_{U_\mu}(f), \qquad (p \geq 1).$$

2. Richter & Sundberg,

$$\mathcal{D}_{P_\nu}(f \wedge f^p) \leq p^2 \mathcal{D}_{P_\nu}(f), \quad (p = 2^k,\ k \geq 1).$$

Put together these inequalities,

$$\mathcal{D}_\omega(f \wedge f^p) \leq p^2 \mathcal{D}_\omega(f), \qquad (p \geq 1).$$

Let $\alpha \in (0,1)$ and let ω be a nonnegative superharmonic weight on $\mathbb{D}$.

Theorem ([8])

Let $f \in \mathcal{D}_\omega \cap \lambda_\alpha$ be a function that vanishes on a closed subset E of $\mathbb{T}$. Then, given $M > 0$, there exists $(f_n)_n \subset \mathcal{D}_\omega \cap \lambda_\alpha$ such that

1. f_n vanishes on E for all $n \geq 1$.
2. $|f_n(z)| = O(dist^M(z,E))$ for all $n \geq 1$.
3. $\lim\limits_{n\to 0} \|f_n f - f\|_{\mathcal{D}_\omega \cap \lambda_\alpha} = 0$.

- M. Matheson : $X = \lambda_\alpha$, Michigan Math. J. (1978).
- B. Bouya : $X = \mathcal{D} \cap \lambda_\alpha$, Canad. J. Math.(2009).

8. H. Bahajji-El Idrissi and O. El-Fallah, Approximation in spaces of analytic functions. *Studia Mathematica, 2019.*

Let $\alpha \in (0,1)$ and let ω be a nonnegative superharmonic weight on $\mathbb{D}$.

- $X = \mathcal{D}_\omega \cap \lambda_\alpha$

$$\mathcal{D}_\omega \cap \lambda_\alpha := \{f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_{\mathcal{D}_\omega \cap \lambda_\alpha} = \|f\|_{\lambda_\alpha} + \mathcal{D}_\omega^{1/2}(f) < \infty\}.$$

The space $\mathcal{D}_\omega \cap \lambda_\alpha$ equipped with the norm $\|.\|_{\mathcal{D}_\omega \cap \lambda_\alpha}$ is a Banach algebra.

Theorem ([9])

Let $\mathcal{I}$ be a non trivial closed ideal of $\mathcal{D}_\omega \cap \lambda_\alpha$. Then

$$\mathcal{I} = \left\{f \in \mathcal{D}_\omega \cap \lambda_\alpha \ : \ f_{|E_\mathcal{I}} = 0, \text{ and } f \in \theta_\mathcal{I} \mathcal{D}_\omega \cap \lambda_\alpha \right\}.$$

- M. Matheson : $X = \lambda_\alpha$, Michigan Math. J. (1978).
- B. Bouya : $X = \mathcal{D} \cap \lambda_\alpha$, Canad. J. Math.(2009).

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9. H. Bahajji-El Idrissi and O. El-Fallah, Approximation in spaces of analytic functions. *Studia Mathematica, 2019.*

Thank you for your attention

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