Kudla Rapoport conjecture over the ramified primes

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Unitary RZ spaces

- $p \geq 3$ prime number. $F/\mathbb{Q}_p$ is a quadratic extension of p-adic fields. We assume π is uniformizer of F. For the moment, we assume the extension is unramified.
- Let $\breve{F}$ be the completion of the maximal unramified extension of F.
- Framing object $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$.
  - $\mathbb{X}$ is a supersingular p-divisible group over $\bar{\mathbb{F}}_p$.
  - $\iota_{\mathbb{X}} : \mathcal{O}_F \to \mathrm{End}(\mathbb{X})$ satisfies signature condition $(n-1, 1)$:

    $$\mathrm{charpol}(\iota_{\mathbb{X}}(a) \mid \mathrm{Lie}\,\mathbb{X}) = (x - \varphi(a))^{n-1}(x - \varphi(\bar{a})),$$

    where $\varphi : \mathcal{O}_{\mathbf{k}} \to S$ is the structure homomorphism.
  - $\lambda_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}^{\vee}$ is a principal polarization whose Rosati involution restricted on $\mathcal{O}_F$ is the Galois conjugation.

Unitary RZ spaces (continued)

- The Rapoport-Zink space $\mathcal{N}_n$ is the formal scheme over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ such that $\mathcal{N}_n(S)$ is the groupoid of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$, where
  - X is a supersingular p-divisible group over S.
  - $\iota : \mathcal{O}_F \to \mathrm{End}(X)$ satisfies signature condition $(n-1, 1)$.
  - $\lambda : X \to X^\vee$ is a compatible principal polarization.
  - $\rho : X \times_S \bar{S} \to \mathbb{X} \times_{\mathrm{Spec}\,\bar{\mathbb{F}}_p} \bar{S}$ is a framing which is a $\mathcal{O}_F$-linear quasi isogeny. We require the height of ρ to be 0.
- p-adic uniformization of the supersingular locus:

$$\widehat{\mathcal{M}}^{ss,p} = \bigsqcup_i \Gamma_i \backslash \mathcal{N}_n.$$

Here $\widehat{\mathcal{M}}^{ss,p}$ is the completion of a unitary Shimura variety along its supersingular locus in characteristic p.

Special cycles

- Let $\mathbb{Y}$ be the unique supersingular p-divisible group of dimension 1, i.e. the underlying p-divisible group of a supersingular elliptic curve. Fix $\iota_0 : \mathcal{O}_F \to \mathrm{End}(\mathbb{Y})$ and $\lambda_0 : \mathbb{Y} \to \mathbb{Y}^\vee$. $(\mathbb{Y}, \iota_0, \lambda_0)$ has a universal deformation $\mathcal{Y}$ which is the canonical lifting of Lubin-Tate, Gross.
- Let $\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$, the set of $\mathcal{O}_F$ quasi-homomorphisms from $\mathbb{Y}$ to $\mathbb{X}$. Then $\mathbb{V} \cong F^n$.
- $\mathbb{V}$ is equipped with a hermitian form $(,)$ defined by

$$(x, y) = \lambda_0^{-1} \circ \check{y} \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}_{\mathcal{O}_F}(\mathbb{Y}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong F.$$

- For $x \in \mathbb{V}$, $\mathcal{Z}(x) \subset \mathcal{N}_n$ is the sub formal scheme where $\rho^{-1} \circ x$ deforms to a homomorphism

$$\mathcal{Y} \to X.$$

  It is nonempty if and only if $(x, x) \in \mathbb{Z}_p$. $\mathcal{Z}(x)$ is called a KR divisor.

Local density

- $F/\mathbb{Q}_p$ is a quadratic extension of p-adic fields. For $\mathcal{O}_F$-Hermitian lattices L and M, define Rep to be the scheme of representations of L by M, for an $\mathbb{Z}_p$ algebra R, we have
$$\mathrm{Rep}(R) = \mathrm{Herm}(L \otimes_{\mathbb{Z}_p} R, M \otimes_{\mathbb{Z}_p} R).$$
- Local density $\alpha(L, M)$:
$$\alpha(L, M) = \lim_{N \to \infty} \frac{\sharp \mathrm{Rep}(\mathbb{Z}_p/p^N)}{p^{N \cdot \dim(\mathrm{Rep})}}.$$
- Let H be the Hermitian lattice represented by
$$\begin{pmatrix} 0 & \frac{1}{\sqrt{\Delta}} \\ -\frac{1}{\sqrt{\Delta}} & 0 \end{pmatrix} (\equiv I_2 \text{ if } p \text{ is inert.})$$
Then
$$\alpha(L, M \oplus H^r) = \alpha(L, M, X)|_{X=p^{-2r}}$$
where $\alpha(L, M, X)$ is a polynomial with $\mathbb{Q}$ coefficients. $\alpha'(L, M)$ is the negative derivative of this polynomial evaluated at $X = 1$.

Kudla-Rapoport conjecture

- Let $L = \mathrm{span}_{\mathcal{O}_F}\{x_1, \dots, x_n\} \subset \mathbb{V}$. Assume that L has rank n.
- $\mathcal{Z}(L)$ is properly supported on $\mathcal{N}_n^{red}$ but in general has dimension bigger than 0. One define

  $$\mathrm{Int}(L) = \langle \mathcal{Z}(x_1), \dots, \mathcal{Z}(x_n) \rangle = \chi(\mathcal{N}_n, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}).$$

  What is behind this notation is that $\mathrm{Int}(L)$ depends only on L (Terstiege, Howard).
- Conjecture (Kudla-Rapoport 2000s, Unramfied prime and principal polarization)
  Assume p is inert and L has rank n. Then

  $$\mathrm{Int}(L) = \partial Den(L) := \frac{\alpha'(L, I_n)}{\alpha(I_n, I_n)}$$

  where I_n is the unique self-dual lattice of rank n, opposite of $\mathbb{V}$.

Global motivation

- The local conjecture has a global counterpart by Kudla which relates the following arithmetic theta series to central derivative of incoherent Eisenstein series:

$$\theta(\tau) = \sum_{T \in \mathrm{Herm}_n(\mathbb{Z})} {}^{\mathbb{L}}\mathcal{Z}(T) \cdot q^T, q^T = \exp(2\pi i \mathrm{tr}(\tau T)),$$

  where ${}^{\mathbb{L}}\mathcal{Z}(T)$ is a (derived) special cycle on an integral model of a unitary Shimura variety.
- Three generations of theta series:
  - Classical (Jacobi, Siegel, Weil);
  - Geometric (Kudla-Millson);
  - Arithmetic theta series (Kudla, etc).

  See `https://www.youtube.com/watch?v=XKenI-Y_HAM` (or just google arithmetic theta series).

Current status and application

- The cases $n = 1, 2$ or more generally the non-degenerate intersection case are checked by Kudla and Rapoport.
- The case $n = 3$ is proved by Terstiege. Already very technical.
- The general case is proved by Li ang Zhang using an ingenious induction.
- One application: The arithmetic inner product formula. Hence the KR conjecture is related to BSD conjecture and Beilinson-Bloch conjecture.

Complication when p is ramified

- When p is ramified. There are two well-understood models of Shimura variety.

  $$\text{Exotic smooth: } \mathrm{Ker}\lambda = \mathrm{Ker}\ \iota(\pi).$$

  $$\text{Krämer model: } \mathrm{Ker}\lambda = \{0\}.$$

  The exotic smooth case is studied by Li and Liu, the analogue of KR conjecture can be proved using the same strategy as Li and Zhang. In this talk, we focus on the Krämer model which only has semi-stable reduction.
- On the analytic side, one has to add correction terms to the derivative of local density in the Krämer model case.

RZ space over a ramified prime I

- Now assume $F/\mathbb{Q}_p$ is a ramified quadratic extension with uniformizer π. $\breve{F}$ is the completion of the maximal unramified extension of F.
- The space $\mathcal{N}_n^{\mathrm{Pap}}$ is the formal scheme over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ such that $\mathcal{N}_n^{\mathrm{Pap}}(S)$ is the groupoid of isomorphism classes of quadruples $(X, \iota, \lambda, \rho)$ over S such that $(X, \iota, \lambda, \rho) \in \mathcal{N}_n^{\mathrm{Pap}}(S)$ (the naive Rapoport Zink space) with the signature condition
  - $\mathrm{char}(\iota(a)|\mathrm{Lie}\,X) = (T - \varphi(a))^{n-1}(T - \varphi(a^\sigma))$ for any $a \in \mathcal{O}_F$.
  - Pappas: $\bigwedge^2(\iota(\pi) - \varphi(\pi)|\mathrm{Lie}\,X) = 0$ and $\bigwedge^n(\iota(\pi) + \varphi(\pi)|\mathrm{Lie}\,X) = 0$.

  Here $\varphi : \mathcal{O}_{\breve{F}} \to \mathcal{O}_S$ is the structural morphism. The space depends on the framing object $\mathbb{X}$.
- $\mathcal{N}_n^{\mathrm{Pap}}$ is the flat closure of the naive moduli functor of Rapoport and Zink, a special case of Pappas-Zhu local model.

RZ space over a ramified prime II

- $\mathcal{N}_n^{\mathrm{Pap}}$ is flat of relative dimension $n-1$ over $\mathrm{Spf}\,\mathcal{O}_{\breve{F}}$ with isolated singularities (the super special points). Blow up these singularities we get $\mathcal{N}_n^{\mathrm{Kra}} \to \mathcal{N}_n^{\mathrm{Pap}}$ (the Krämer model). The super special points becomes exceptional divisors isomorphic to $\mathbb{P}^{n-1}/\mathrm{Spec}\,\bar{\mathbb{F}}_p$.
- Why is the Krämer model a good one? We can think of it as a "minimal model" of our Shimura variety (RZ space) over $\mathbb{Z}_p$, regular and of semi-stable reduction. In fact, when $n=2$, it is a minimal arithmetic surface. Like the Pappas model, the Krämer model has a nice moduli interpretation.
- For each even n there are two non isomorphic $\mathcal{N}_n^{\mathrm{Kra}}$, while for each odd n, $\mathcal{N}_n^{\mathrm{Kra}}$ is unique. This corresponds to the similarity classes of Hermitian forms.

Special cycles in the Krämer model

- Define $\mathbb{V} = \mathrm{Hom}_{\mathcal{O}_F}(\mathbb{Y}, \mathbb{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L = \mathrm{span}_{\mathcal{O}_F}\{x_1, \ldots, x_n\} \subset \mathbb{V}$ as before.
- For $x \in \mathbb{V}$, $\mathcal{Z}^{\mathrm{Kra}}(x) \subset \mathcal{N}_n^{\mathrm{Kra}}$ is the sub formal scheme where $\rho^{-1} \circ x$ deforms to a homomorphism. It is nonempty if and only if $(x, x) \in \mathbb{Z}_p$. $\mathcal{Z}^{\mathrm{Kra}}(x)$ is a divisor (Howard). Similarly define $\mathcal{Z}^{\mathrm{Pap}}(x)$.
- Payoff: $\mathcal{Z}^{\mathrm{Kra}}(x)$ always contains some exceptional divisors. So their intersections if nonempty will never have zero dimension. Define $\mathrm{Int}(L)$ as before using derived intersection.

$$\mathrm{Int}(L) = \langle \mathcal{Z}(x_1), \ldots, \mathcal{Z}(x_n) \rangle = \chi(\mathcal{N}_n, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}).$$

First guess of the conjecture

- Take M to be the unique unimodular hermitian lattice of rank n whose invariant is the opposite of $\mathbb{V}$ (hence L). The unmodified KR conjecture would look like

$$\mathrm{Int}(L) = 2\frac{\alpha'(L, M)}{\alpha(M, M)}.$$

- A simple observation: when L is not integral, i.e. the hermitian form on L does not take values in $\mathcal{O}_F$, $\mathcal{Z}^{\mathrm{Kra}}(L)$ is empty. This implies $\mathrm{Int}(L) = 0$.
- However the local density polynomial $\alpha(L, M, X)$ is not zero where

$$\alpha(L, M \oplus H^r) = \alpha(L, M, X)|_{X=p^{-2r}}$$

since H is not integral. Hence the unmodified KR conjecture can not hold.

Correction term on the analytic side II

- Consider a linear combination:

$$2\frac{\alpha'(L,M)}{\alpha(M,M)}+\sum_{i=1}^{n-1}c_i\alpha(L,M_i)$$

where M_i is of the form

$$M_i = H^{\lfloor \frac{i+1}{2} \rfloor} \oplus \text{a unimodular hermitian lattice of rank } n-2\lfloor\frac{i+1}{2}\rfloor.$$

Notice that we have exhausted all such choices up to equivalence.

- Solve the linear system in $(c_i)_{1\le i\le n-1}$

$$2\frac{\alpha'(M_j,M)}{\alpha(M,M)}+\sum_{i=1}^{n-1}c_i\alpha(M_j,M_i)=0$$

for $1\le j\le n-1$. Notice that the matrix $(\alpha(M_j,M_i))_{1\le i,j\le n-1}$ is upper trangular with nonzero entries.

The modified conjecture

- Conjecture (He-Shi-Yang 2021): Assume that M is unimodular of rank n whose invariant is the opposite of $\mathbb{V}$, then
$$\mathrm{Int}(L) = 2\frac{\alpha'(L, M)}{\alpha(M, M)} + \sum_{i=1}^{n-1} c_i \alpha(L, M_i).$$
- Main results up to now:
  - $n = 1$, exercise.
  - When $n = 2$ and $\mathbb{V}$ is anisotropic (Shi 2020).
  - When $n = 2$ and $\mathbb{V}$ is isotropic (He-Shi-Yang 2020).
  - When $n = 3$ (He-Shi-Yang 2021).

Proof when $n = 2$

- The proof is computing both sides of the equation explicitly. On the arithmetic-geometric side, use low-dimensional exceptional isomorphism between moduli spaces.
- When $\mathbb{V}$ is anisotropic, $\mathcal{N}_2^{\mathrm{Pap}}$ is isomorphic to $\mathcal{M}_{\Gamma_0(p)}$ (Rapoport-Smithling-Zhang). The proof of Theorem A uses quasi-canonical liftings of Gross. When $\mathbb{V}$ is isotropic. $\mathcal{N}_2^{\mathrm{Pap}}$ is isomorphic to $\check{\Omega}$, the formal completion of Drinfeld p-adic half plane. The proof of Theorem B uses Grothendieck-Messing theory to write down the equations of special cycles explicitly.

Bruhat-Tits building

- For an $\mathcal{O}_F$ lattice $\Lambda \subset \mathbb{V}$, define its dual

$$\Lambda^\sharp = \{x \in \mathbb{V} \mid (x, y) \in \mathcal{O}_F, \forall y \in \Lambda\}.$$

  We say a lattice is a vertex lattice if $\pi\Lambda \subseteq \Lambda^\sharp \subseteq \Lambda$. $t(\Lambda) = \dim_{\mathbb{F}_p}(\Lambda/\Lambda^\sharp)$ is called the type of Λ. t must be even by principal polarization.
- When $n = 3$: a vertex lattice is of type 0 or 2. Every type 2 lattice contains exactly $p + 1$ type 0 lattices. Every type 0 lattice is contained in exactly $p + 1$ type 2 lattices.

Special fiber of $\mathcal{N}_3$

- The reduced locus of $\mathcal{N}_3^{\mathrm{Pap}}$ is "dual" to the above Bruhat-Tits tree: Every type 2 lattice Λ_2 corresponds to a projective line $\mathcal{V}_{\Lambda_2} = \mathbb{P}(\Lambda_2/\Lambda_2^\sharp)$. $\mathcal{V}_{\Lambda_2}$ contains $p+1$ super special points each corresponding to a type 0 lattice. Each super special point has $p+1$ projective lines $\mathcal{V}_{\Lambda_2}$ passing through it.
- The blow up $\mathcal{N}_3^{\mathrm{Kra}} \to \mathcal{N}_3^{\mathrm{Pap}}$ creates an exceptional divisor $\mathrm{Exc}_{\Lambda_0} = \mathbb{P}^2$ for each super special point. We also choose a hyperplane in Exc_{Λ_0} and denote it by H_{Λ_0}.

Some basic facts (for general n)

- Let $\tilde{\mathcal{Z}}(x)$ be the strict transform of $\mathcal{Z}^{\mathrm{Pap}}(x)$ w.r.t. the blow-up $\mathcal{N}_n^{\mathrm{Kra}} \to \mathcal{N}_n^{\mathrm{Pap}}$. When (x, x) is in $\mathbb{Z}_p^{\times}$, $\tilde{\mathcal{Z}}(x) \cong \mathcal{N}_{n-1}^{\mathrm{Kra}}$. In other words, there is a closed embedding $\mathcal{N}_{n-1}^{\mathrm{Kra}} \hookrightarrow \mathcal{N}_n^{\mathrm{Kra}}$.
- Let Λ be a type 0 lattice and $m_\Lambda(x)$ be the largest integer m such that $\pi^{-m} \cdot x \in \Lambda$. Then

$$\mathcal{Z}^{\mathrm{Kra}}(x) = \tilde{\mathcal{Z}}(x) + \sum_{\substack{\Lambda \text{ is a type 0 lattice,} \\ x \in \Lambda}} (m_\Lambda(x) + 1)\mathrm{Exc}_\Lambda,$$

and

$$[\tilde{\mathcal{Z}}(x) \cap \mathrm{Exc}_\Lambda] = (2m_\Lambda(x) + 1)H_\Lambda.$$

Difference cycle

- For a lattice $L = \mathrm{span}\{x_1, \ldots, x_m\} \subset \mathbb{V}$, define

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) := [\mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}^{\mathrm{Kra}}(x_m)}] \in K_0(\mathcal{N}_n^{\mathrm{Kra}})$$

  where $[F] = \sum_i (-1)^i [H_i(F)]$ for a complex F of coherent sheaves.
- Define $^{\mathbb{L}}\mathcal{D}^{\mathrm{Kra}}(L)$ by

$$^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) + \sum_{i=1}^{m} (-1)^i p^{i(i-1)/2} \sum_{g \in U_m \backslash \pi_{m,i}} {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L[g^{-1}])$$

  where $U_m = \mathrm{GL}_m(\mathcal{O}_F)$ and $\pi_{m,i} = U_m(\pi I_i \oplus I_{m-i})U_m$.
- When $m = 1$,

$$^{\mathbb{L}}\mathcal{D}^{\mathrm{Kra}}(x) = \mathcal{Z}^{\mathrm{Kra}}(x) - \mathcal{Z}^{\mathrm{Kra}}(x/\pi)$$

  is the difference divisor studied by Terstiege.

Difference cycle in the three dimension case

- The most interesting case to us is when $m = n - 1$.
- When $n = 3$, $m = 2$, we have

$$^{\mathbb{L}}\mathcal{D}^{\mathrm{Kra}}(L) = {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L) - \sum_{L \subset L' \subset \frac{1}{\pi}L} {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(L') + p \cdot {}^{\mathbb{L}}\mathcal{Z}^{\mathrm{Kra}}(\frac{1}{\pi}L).$$

- An example in the above case. Suppose $\pi | (L, L)$, then

$$^{\mathbb{L}}\mathcal{D}^{\mathrm{Kra}}(L) = \sum_{\Lambda_2 \in \mathcal{T}(L)} (2\mathcal{V}_{\Lambda_2} + \sum_{\Lambda_0 \subset \Lambda_2} H_{\Lambda_0})$$

  where $\Lambda_2 \in \mathcal{T}(L)$ iff $\mathcal{V}_{\Lambda_2} \subset \mathcal{Z}^{\mathrm{Kra}}(L)$. We prove the above by intersecting ${}^{\mathbb{L}}\mathcal{D}^{\mathrm{Kra}}(L)$ with $\tilde{\mathcal{Z}}(y_0)$ where $\{y_0\}^{\perp}$ is anisotropic. Notice that in this case all horizontal cycles cancel out with each other by a version of inclusion-exclusion principle.

Proof by induction on the valuation

We can then use induction on the valuation of L to prove the conjecture. The induction starts from the basic cases which essentially reduces to the case when $n = 2$. Then it suffices to show

$$^{\mathbb{L}}\mathcal{D}^{\mathrm{Kra}}(L) \cdot \mathcal{Z}^{\mathrm{Kra}}(x) = \beta(L \oplus \{x\}, M)^{(2)}$$

where $\beta(L \oplus \{x\}, M)^{(2)}$ is certain primitive local density. It is equal to (Kitaoka, Katsurada)

$$\alpha(L \oplus \{x\}, M) + \sum_{i=1}^{m} (-1)^i p^{i(i-1)/2} \sum_{g \in U_m \backslash \pi_{m,i}} \alpha(L[g^{-1}] \oplus \{x\}, M).$$

In fact this motivates our definition of difference cycle.

Failure of local modularity

For $n \in \mathbb{Z}_{\geq 2}$.

- A function $f \in S(\mathbb{V})$ is modular if
$$f(x) = -\hat{f}(x).$$
- If Λ_2 is a type 2 lattice, define
$$\mathrm{Int}_{\Lambda_2}(x) := \langle \mathcal{V}_{\Lambda_2}, \mathcal{Z}^{\mathrm{Kra}}(x) \rangle.$$
Then
$$\mathrm{Int}_{\Lambda_2}(x) = 1_{\Lambda_2}(x).$$
- If Λ_0 is a type 0 lattice, define
$$\mathrm{Int}_{\Lambda_0}(x) := \langle \mathbb{P}^1_{\Lambda_0}, \mathcal{Z}^{\mathrm{Kra}}(x) \rangle$$
where $\mathbb{P}^1_{\Lambda_0}$ is any projective line in Exc_{Λ_0}. Then
$$\mathrm{Int}_{\Lambda_0}(x) = -1_{\Lambda_0}(x).$$
- When $n = 2$. One of $1_{\Lambda_2}(x)$ and $1_{\Lambda_0}(x)$ is modular and the other is not.

Final remark

Due to the failure of local modularity, the method of Li and Zhang in the good reduction case does not (directly) apply to our situation. So the general case of the conjecture seems out of reach at this point.

Happy birthday, Steve!