Zagier's Polylogarithm Conjecture revisited

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The origins

- Dirichlet's (analytic) Class Number Formula for F a number field (with number ring $\mathcal{O}_F$).

Peter Gustav Lejeune Dirichlet

Relates certain fundamental arithmetic invariants—(ideal) class number h_F, units $\mathcal{O}_F^\times$ and number of roots of unity $w_1(F)$—to one single "special value".
More precisely, define the Dedekind zeta function of F (denoting $N(I)$=ideal norm of I) $\zeta_F(s) = \sum_{\mathcal{A}\neq(0)} N(\mathcal{A})^{-s}$, $\Re(s) > 1$, and "evaluate" at $s = 1$, or via fctl. eqn. relating $s \leftrightarrow 1-s$, $\zeta_F^*(0)$(=first non-vanishing coeff. in the Taylor exp. at $s = 0$).

$$\boxed{\zeta_F^*(0) = -\frac{h_F\, R_1(F)}{w_1(F)}},$$

where $R_1(F) = \mathrm{covol}(\mathcal{O}_F^\times)$ denotes the Dirichlet regulator of F, i.e. the covolume of the lattice spanned by a set of fundamental units of $\mathcal{O}_F^\times$ via $(\log|\sigma(\cdot)|)_\sigma$, $\sigma : F \hookrightarrow \mathbf{C}$.

An analogue

Do other evaluations, say at $s = 1 - m$, allow for a similar formula—and arithmetic interpretation?

Q. Roughly, is there a relationship, for $m > 1$,

$$\zeta_F^*(1-m) \sim (\text{higher class no})(\text{higher regulator}),$$

where $\sim$ still requires specification?

A. Seminal work by Borel ('74, '78) indeed established

- the rank of all relevant algebraic K-groups, and defined
- "higher regulators" arising from the (odd-indexed) groups.
- Furthermore, he related the m-th regulator to $\zeta_F^*(1-m)$.

In a nutshell: The rank is 0 for $K_{2i}(\mathcal{O}_F)$, $i > 0$, and

$$d_m := \operatorname{rk} K_{2m-1}(\mathcal{O}_F) = \begin{cases} r_2 & \text{if } m > 1 \text{ even}, \\ r_1 + r_2 & \text{if } m > 1 \text{ odd}. \end{cases}$$

Lichtenbaum's question

Lichtenbaum subsequently(!) in '72 gave an interpretation of what should be considered as a "higher class number": the order of a specific (even-indexed) higher algebraic K-group. Hence, up to for $m > 1$, he expected

$$\zeta_F^*(1-m) \sim (m\text{-th higher class no})(m\text{-th Borel regulator}),$$

where $\sim$ denotes "up to known factors". How does that reconcile with Dirichlet's CNF? Turns out:

- $K_0(\mathcal{O}_F) = \mathbf{Z} \oplus \mathcal{C}\ell(\mathcal{O}_F)$, so $h_K = |\mathcal{C}\ell(\mathcal{O}_F)| = |K_0(\mathcal{O}_F)_{\mathrm{tors}}|$,
- $K_1(\mathcal{O}_F) = \mathcal{O}_F^\times$, hence $R_1(F) = \mathrm{covol}(K_1(\mathcal{O}_F))$.

Lichtenbaum's analogue, rough form (and the original):

$$\zeta_F^*(1-m) \sim |K_{2m-2}(\mathcal{O}_F)|\mathrm{covol}(K_{2m-1}(\mathcal{O}_F/\mathrm{tors})),$$

Question 4.2. When is it true that

$$\lim_{s\to -m} \zeta(F,s)(s+m)^{-g} = \pm \frac{\# K_{2m}(\mathcal{O}_F)}{\# K_{2m+1}(\mathcal{O}_F)_{\mathrm{tor}}} \cdot R_m(F)?$$

Bloch's pioneering work

Drawback: Highly abstract definition of both K-groups $K_i(F)$, for $i > 2$, and the associated higher regulators. Even to date, only calculation via original def. has been done in case $F = \mathbf{Q}(\sqrt{-3})$ (Choo, Mannan, Sanchez-Garcia, Snaith, 2012), but only a bounded interval could be given.

Enter Bloch (Steele Prize 2021): Famous Irvine lecture notes (from '78, published in '00(!)) established a rather explicit candidate for $K_3(F)$. Moreover, he suggested an associated regulator map (originally studied by D.Wigner).

Bloch's crucial idea: Study the kernel of the following map

$$\begin{aligned} \delta_2^F : \mathbf{Z}[F] &\to \textstyle\bigwedge^2 F^\times \\ [x] &\mapsto x \wedge (1-x), \end{aligned}$$

and factor out "universal relations" (independent of F).

Expectation: $B_2(F) = \frac{\ker \delta_2^F}{\langle \text{univ.rel.} \rangle}$ ("Bloch gp") is close to $K_3(F)$.

Suslin's confirmation, and elements in $\ker \delta_2^F$

Suslin: Established a precise relationship (short exact sequence $0 \to T(F) \to K_3(F)/K_3^M(F) \to B_2(F) \to 0$, for some torsion groups $T(F)$ and $K_3^M(F) =$'Milnor K-group').
In particular $K_3(F)$ and $B_2(F)$ indeed have the same rank.

A few elements in $\ker \delta_2^F$

- $[\frac{1}{2}] \in \ker \delta_2^{\mathbf{Q}}$;
- $[x] + [1-x], \quad 2\big([x] + [1/x]\big) \in \ker \delta_2^{\mathbf{Q}(x)}$;
- $[9] - 6[3] \in \ker \delta_2^{\mathbf{Q}}$;
- In the field $F = \mathbf{Q}(\sqrt{-7})$ the number $\alpha = \frac{1+\sqrt{-7}}{2}$, root of $x^2 - x + 2$, satisfies $\alpha^2 + (1-\alpha) = -1$, so $1 + \frac{1-\alpha}{\alpha^2} = \frac{-1}{\alpha^2}$. Therefore, using $a \wedge bc = a \wedge b + a \wedge c$ and working modulo 2-torsion (we can ignore signs in each wedge factor), we get

$$\delta_2^F\left([-\frac{1-\alpha}{\alpha^2}]\right) = (1-\alpha) \wedge \frac{-1}{\alpha^2} + \frac{-1}{\alpha^2} \wedge \frac{-1}{\alpha^2} \equiv -2\big((1-\alpha) \wedge \alpha\big),$$

so $2[-\frac{1-\alpha}{\alpha^2}] + [\alpha]$ is in $\ker \delta_2^F$ (and $\mathcal{L}_2$ maps it to $\sim \zeta_F^*(-1)$).

Universal relations and $\mathrm{Li}_2(z)$

Universal: Let $z_i \in F$ $(i = 1, \dots, 5)$ satisfy $1 - z_i = z_{i-2} z_{i+2}$ (indices mod 5), then $\sum_{i=1}^{5} [z_i] \in \ker \delta_2^F$ ("five term relation").

The Bloch-Wigner dilogarithm. A function that **vanishes** on these universal relations is given by a single-valued variant of Euler's dilogarithm $\mathrm{Li}_2(z) = \sum_{n \geq 1} z^n / n^2$, originally studied by D.Wigner and employed in Bloch's lecture notes, is

$$\mathcal{L}_2(z) = \Im\big(\mathrm{Li}_2(z) - \log|z| \mathrm{Li}_1(z)\big) .$$

Hence $\mathcal{L}_2(z)$ can be extended to a function on the quotient $B_2(F)$, for $F \hookrightarrow \mathbf{C}$, and thus might play the role of a higher regulator for the group $K_3(F)$. Indeed, giving credit to Goncharov for ε,

Theorem. (Zagier '85, Borel+Bloch+Beilinson+Suslin+ε)
For a number field F $\exists \xi_1, \dots, \xi_{r_2} \in \ker \delta_2^F$ and $q \in \mathbf{Q}^\times$ s.t.

$$\pi^{r_2} \zeta_F^*(-1) = q \cdot \det \Big(\mathcal{L}_2(\xi_i^\sigma) \Big)_{i,\sigma} .$$

In other words, ZPC holds for $m = 2$.

More precise version à la Lichtenbaum, for *abelian* fields, shown ('95-'03) by Kolster-Nguyen Quang Do-Fleckinger (+Benois), by Huber-Kings, and also by Burns-Greither; all up to a power of 2.

Recent update (jt. + Burns, de Jeu, Rahm, Yasaki, '21): This power of 2 can be settled, using work of Flach, allows to derive the first genuine generators of $K_3(\mathcal{O}_F)$ for some imag. quadratic fields.

Example. For $F = \mathbf{Q}(\sqrt{-4547})$ the following element generates $K_3(\mathcal{O}_F)$ is a generator for the "torsion free part" ($\omega = \frac{1+\sqrt{-4547}}{2}$). Moreover, this implies that $K_2(\mathcal{O}_F)$ has order 233 (a prime).

$$
\begin{aligned}
&132[\tfrac{-2\omega+5}{117}] - 2[\tfrac{-1}{12}] + 8[\tfrac{-2\omega-3}{1404}] - 6[\tfrac{-2\omega+5}{18}] - 14[\tfrac{-2\omega+1752}{19683}] - 2[\tfrac{-1}{2}] + 8[\tfrac{-2\omega-3}{27}] + 2[\tfrac{-1}{288}] + 2[\tfrac{-1}{3}] - 24[\tfrac{-2\omega+1752}{3159}] - 2[\tfrac{-1}{36}] + 128[\tfrac{-2\omega+5}{39}] \\
&+ 24[\tfrac{-2\omega-3}{4212}] + 74[\tfrac{-2\omega+5}{52}] + 12[\tfrac{-2\omega+5}{54}] - 12[\tfrac{-2\omega-840}{6591}] - 54[\tfrac{-\omega+421}{351}] + 40[\tfrac{-2\omega-3}{72}] - 16[\tfrac{-2\omega-3}{78}] + 12[\tfrac{-2\omega+5}{8}] - 10[\tfrac{-\omega+421}{468}] + 2[\tfrac{-13}{16}] + 4[\tfrac{-13}{18}] \\
&- 6[\tfrac{-13}{24}] + 2[\tfrac{-13}{243}] - 2[\tfrac{-13}{3}] - 2[\tfrac{-13}{4}] - 8[\tfrac{-169}{324}] - 2[\tfrac{-4\omega-1680}{177957}] - 6[\tfrac{-208}{81}] - 2[-26] - 4[\tfrac{-26}{3}] + 2[\tfrac{-27}{4}] + 42[\tfrac{-3\omega+1263}{2197}] - 12[\tfrac{-31\omega+162}{13182}] \\
&+ 22[\tfrac{-31\omega-131}{2197}] - 4[\tfrac{-5\omega+6}{64}] - 8[\tfrac{-16\omega+6736}{2197}] + 2[\tfrac{-16\omega-4523}{2197}] - 26[\tfrac{2\omega-5}{1404}] + 2[\tfrac{2\omega+3}{18}] + 2[\tfrac{1}{18}] - 4[\tfrac{\omega+875}{1053}] - 4[\tfrac{\omega+875}{1296}] - 24[\tfrac{2\omega-5}{27}] + 12[\tfrac{2\omega+1750}{3159}] \\
&- 2[\tfrac{1}{32}] + 14[\tfrac{2\omega-5}{351}] - 14[\tfrac{2\omega-5}{4212}] - 50[\tfrac{2\omega+3}{52}] - 78[\tfrac{2\omega+3}{54}] + 4[\tfrac{2\omega-842}{6591}] + 38[\tfrac{\omega+420}{351}] - 30[\tfrac{2\omega-5}{72}] - 2[\tfrac{2\omega-5}{78}] - 14[\tfrac{2\omega+3}{8}] + 14[\tfrac{\omega+420}{4056}] - 6[\tfrac{\omega+420}{468}] \\
&- 6[117] - 2[\tfrac{13}{256}] - 2[\tfrac{13}{81}] - 4[\tfrac{169}{16}] - 14[\tfrac{169}{243}] + 4[\tfrac{169}{256}] - 10[\tfrac{3494\omega-13298}{2197}] - 16[\tfrac{4\omega-1684}{177957}] - 42[\tfrac{16\omega+4523}{8788}] + 2[\tfrac{243}{256}] + 2[\tfrac{26}{27}] - 2[\tfrac{26}{9}] + 4[\tfrac{3}{32}] \\
&- 16[\tfrac{3\omega+1260}{2197}] + 10[\tfrac{31\omega+131}{13182}] - 24[\tfrac{31\omega-162}{2197}] - 6[\tfrac{39}{2}] + 6[\tfrac{39}{8}] + 8[\tfrac{8\omega+3360}{351}] + 20[\tfrac{8\omega+3360}{9477}] - 30[\tfrac{10\omega+2}{1053}] - 38[\tfrac{5\omega+1}{54}] + 6[\tfrac{5\omega+1}{64}] - 12[\tfrac{5\omega-6}{78}] \\
&- 4[\tfrac{5\omega+1}{1053}] + 24[\tfrac{5\omega+1}{27}] + 8[\tfrac{10\omega-12}{729}] - 6[52] - 4[\tfrac{52}{81}] - 2[\tfrac{64}{81}] - 14[\tfrac{16\omega+4523}{4563}] + 4[\tfrac{841\omega-176104}{177957}]
\end{aligned}
$$

Higher weight

Q. How to generalise δ_2, e.g. what is a candidate for $K_5\mathcal{O}_F$?

A. Zagier (late '80s): defined a "higher Bloch group" on which a single-valued version of $\mathrm{Li}_3(z) = \sum_{n\geq 1} z^n/n^3$ is defined and which plays a "regulating" role (i.e. maps to a lattice in some **R**-v.s.).

First approximation: Define analogously the 'fingerprint' map

$$\begin{aligned} \delta_3^F : \mathbf{Z}[F] &\to F^\times \otimes \bigwedge^2 F^\times \\ [x] &\mapsto x \otimes x \wedge (1-x). \end{aligned}$$

This time taking the kernel is not good enough in general, but one needs to impose further conditions. Indeed, any homomorphism $\iota : F^\times \to \mathbf{Z}$ induces a map $\bar\iota : \ker \delta_3^F \to \ker \delta_2^F$ (on the first tensor factor). Demand: to land in the intersection $A_3(F) = \bigcap_\iota \ker \bar\iota$.

Again, one defines the subgroup of "universal elements", say $C_3(F) \subset A_3(F)$, and takes the quotient $B_3(F) = A_3(F)/C_3(F)$. This is Zagier's candidate for $K_5\mathcal{O}_F$ (after $\otimes\mathbf{Q}$).

A function that is defined on the quotient: a single-valued version $\mathcal{L}_3(z)$ of the trilogarithm $\mathrm{Li}_3(z)$ (found, for all $\mathrm{Li}_m(z)$, by Ramakrishnan implicitly and made explicit by both Wojtkowiak and Zagier).

By iterating this process Zagier defined analogues for even higher Bloch groups $B_m(F) = A_m(F)/C_m(F)$, which putatively play the role of $K_{2m-1}(F)$. Then the gist of (a corollary of) his conjecture for 'weight' m can be formulated as follows.

Zagier's Polylogarithm Conjecture. For a number field F and $m > 1$ there are elements $\xi_j \in A_m(F)$, $j = 1, \ldots, d_m$ s.t.

$$\zeta_F^*(1-m) \sim \det\left(\mathcal{L}_m(\xi_j^\sigma)\right)_{j,\sigma}$$

where σ runs through appropriate field embeddings ($\sim$ again indicating "up to a power of π and a rational factor").

Goncharov's program

Goncharov. Ingenious new point of view: Associate 'trilogarithmic values' to configurations of six points in the projective plane.

Thm. (Goncharov, '90) ZPC holds for $m = 3$.

Important new functional equation for the trilogarithm via new projective invariant for a configuration as above, his "triple ratio" $\mathrm{Alt}_6\left(\frac{|124|\cdot|235|\cdot|316|}{|125|\cdot|236|\cdot|314|}\right)$; $\mathcal{L}_3(..)$ represents a cont. 5-cocycle for GL_3.

More conceptual +refined approach to ZPC, invoking graded pieces of K-groups, and a conjectural motivic co-Lie algebra for *any* field, in which his newly defined polylogarithmic/motivic complexes should be contained in a natural way.

- Vast and visionary program
- Suggested many tools to attack ZPC for higher weight m.
- In particular, it spurred the search for higher cross ratios.

Despite tremendous progress (medal at ECM Paris '92!), the same approach for $m = 4$ has not yet been successful.

Multiple polylogarithms

Consider also multi-variable analogues of polylogarithms, so-called *multiple polylogarithms* (MPL's), the simplest new ones being double polylogs, defined by

$$\mathrm{Li}_{a,b}(x,y) = \sum_{m>n>0} \frac{x^m}{m^a} \frac{y^n}{n^b} \,.$$

He singled out one "combinatorial" problem that would go a long way towards solving ZPC for $m = 4$. Write $V(x,y)$ for the formal five term relation in the two variables x, y.

Goncharov's Conjecture: There should be functions $f_i(x,y,z) \in \mathbf{Q}(x,y,z)$ ($i \in I$ finite) such that

$$I_{3,1}(z, V(x,y)) = \sum_i \mathrm{Li}_4\big(f_i(x,y,z)\big)$$

modulo products of lower weight terms.
(Here $I_{3,1}(x,y) = \mathrm{Li}_{3,1}(y/x, 1/y)$, "iterated integral version".)

Goncharov's symbol and Feynman integrals

While solutions to this innocuous looking conjecture could be found for 1-variable functional equations (Zagier '93, G. '00) in place of $V(x, y)$, the original task seemed intractable.

Goncharov kept churning out important results, e.g. he showed

- that a certain multi-variable "Grassmannian" polylog satisfies the sought-for cocycle condition;
- (with Deligne) the existence of said motivic coLie algebra for number fields,
- studied (with Fock) a connection to cluster algebras,
- and in particular he defined a **symbol**, an 'algebraic fingerprint' for the MPL's (generalising the δ_m^F above).

Roughly, a symbol is a sum of elementary tensors which captures the differential behaviour of the associated function, the combinatorics being encoded in terms of trivalent trees.

All these ingredients should eventually culminate in ZPC for $m = 4$.

Furthermore, he collaborated with physicists (Spradlin et al.): Goncharov's motivic insights and ideas helped recast a hideously long expression for an important integral (found by Duhr et al.), the so-called six-point 1-loop remainder function $R_6^{(1)}$ in $\mathcal{N} = 4$ SYM theory. Drastic reduction from 17pp. to essentially one long line (+conventions). Sketchy 5pp. note.

$\rightsquigarrow$ Collaboration with Duhr (+ Rhodes), reverse-engineering their results, culminating in an implementation of the 'symbol' that allowed for experiments. By-product:

Thm. There are 122 rational functions $f_i(x, y, z)$ and integers c_i which solve Goncharov's Conjecture, i.e.

$$I_{3,1}\Big(z, [x]+[y]+\Big[\frac{1-x}{1-xy}\Big]+[1-xy]+\Big[\frac{1-y}{1-xy}\Big]\Big) \equiv \sum_{i=1}^{122} c_i \mathrm{Li}_4\big(f_i(x, y, z)\big)\,,$$

modulo products of lower weight functions.

Found a few days after Zagier's 60th birthday celebrations...

Fast forward to Zagier's 65th birthday celebrations ("Modular forms are everywhere"). Invited talk by Goncharov–announces yet another new point of view on the problem, motivated and guided by cluster algebra complexes, plus a new concept of "motivic correlators" (distinguished elements in motivic coLie algebra).

Goncharov-Rudenko: A conceptual way to derive a crucial new relation denoted $\mathbf{Q}_4$ (relating correlators of type $I_{3,1}$ and Li_4) that plays a role akin to the 122-term one above; the previous triple ratio formula is recast as $\mathbf{Q}_3$ (relating type $I_{2,1}$ and Li_3), (DR assures me that my original identity provided both motivation and inspiration for their deduction.)

Moreover, they also establish the connection to algebraic K-theory to the extent that they can claim

Theorem. (Goncharov-Rudenko, '18) ZPC holds for $m = 4$.

Still lacking in this–still somewhat implicit–approach: an explicit "quadruple ratio".

In joint work with Charlton and Radchenko we could establish this—alas, the current expression is still rather unpleasant.

Theorem. (Cha-Ga-Ra, '19) There is a (complicated) explicit form of the quadruple ratio (= a formal linear combination of rational functions on configurations of 8 points in $\mathbf{P}^3$ whose image under $\mathcal{L}_4$ represents a non-trivial cohomology class in $H^7_*(GL_4(\mathbf{C}), \mathbf{R})$). It comprises 368 orbits under an action of the alternating group on 8 letters.

Further results in higher weight:
Theorem. (Cha-Ga-Ra, '19) There are explicit analogues of $\mathbf{Q}_m$ for $m = 5, 6, 7$.
This could provide an important stepping stone for a solution of ZPC in those weights.

Breaking (recent amazing preprint by Rudenko).
Solves Goncharov's depth reduction conjecture (MPL's in weight m are generated by MPL's in depth at most $m/2$).
Announces similar results to $\mathbf{Q}_m$ above, joint with Matveiakin.

Thank You!