Elliptic surfaces and the enumeration of walks with small steps in the quarter plane

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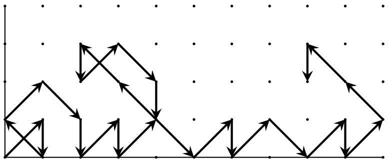
Unweighted walks

Consider walks starting form $(0,0)$ with set of steps in

$$\mathcal{D} \subset \{\leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow\} \leftrightarrow \{(i,j) \mid i,j \in \{-1,0,1\}\}$$

An example

$$\mathcal{D} = \{\nwarrow, \nearrow, \searrow, \downarrow\}.$$



Weighted models and Generating Series

Weighted Model: Assign a set of probabilistic weights to each cardinal direction

$$\mathcal{W} = \{(d_{i,j})_{i,j=-1,0,1} \in (\mathbb{Q} \cap [0,1])^9 \text{ with } \sum d_{i,j} = 1\},$$

associated with a set of directions $\mathcal{D} := \{(i,j) | d_{i,j} \neq 0\}$

Unweighted model: all the non-zero weights are equal and $d_{0,0} = 0$.

Weighted coefficients: Fix a weighted model $\mathcal{W}$ and set $q(l,s,k)$ is the probability for a weighted walk to end at (l,s) in k steps.

Generating Series: Fix $\mathcal{W}$ (and therefore $\mathcal{D}$) and consider

$$Q(x,y,t) = \sum_{l,s,k} q(l,s,k) x^l y^s t^k$$

converges for $|x|, |y| \leq 1$ and $|t| < 1$.

Classification for walks in the quadrant

Combinatorial question:
Fix $\mathcal{W}$ and count the number $q(l, s, k)$ of walks in the first quadrant with steps in $\mathcal{D}$ ending at (l, s) after k steps? Too difficult in general but one can search

- for asymptotics of the sequence $(q(l, s, k))$.
- for an algebraic classification of the associated generating series.

Classification problem: when is $Q(x, y, t)$

- Algebraic over $\mathbb{Q}(x, y, t)$?
- Holonomic over $\mathbb{Q}(x, y, t)$? (x-, y-, and t-holonomic)
- Differentially Algebraic over $\mathbb{Q}(x, y, t)$? (x-,y-, and t-DA)

$f(x, y, t)$ is x-*DA* if for some n and nonzero polynomial P,

$$P(x, y, t, f, \frac{\partial f}{\partial x}, \dots, \frac{\partial^n f}{\partial x^n}) = 0$$

Classification

Inspired by Fayolle, Iasnogorodski, Malyshev (1999) for random walks, Bousquet-Mélou, Mishna (2010) associate to a model $\mathcal{W}$,

- an algebraic curve E of genus 0 or 1, and
- a group G, finite or infinite.

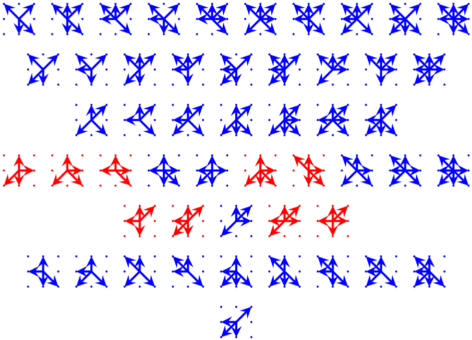
Bousquet-Mélou-Mishna conjecture: $Q(x, y, t)$ holonomic $\Leftrightarrow |G| < \infty$

Unweighted model 256 choices for $\mathcal{D} \xrightarrow{\textit{triviallity}, \textit{symmetries}}$ 79 interesting ones

Results: For the **79** unweighted models

- $|G| < \infty$ for **23** walks $\Rightarrow Q(x, y, t)$ algebraic or holonomic.
  $\rightarrow$ A. Bostan, M. Bousquet-Mélou, M. van Hoeij, M. Kauers, M. Mishna, ...
- $|G| = \infty$ for **56** walks $\Rightarrow Q(x, y, t)$ **not** holonomic.
  - 5 walks with genus$(E) = 0$ $\rightarrow$ S. Melzcer, M. Mishna, A. Rechnitzer, ...
  - 51 walks with genus$(E) = 1$ $\rightarrow$A. Bostan, I. Kurkova, K. Raschel, B. Salvy, ...
- If the series is not holonomic is it differentially Algebraic?

The unweighted 51 models with $|G| = \infty$, genus$(E) = 1$



Theorem (Dreyfus-H.-Roques-Singer, 2018):
For $t \in \mathbb{R} \backslash \overline{\mathbb{Q}}$

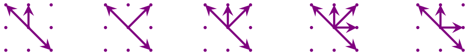
1. In 42 cases, $Q(x, y, t)$ is not x, y-DA.
2. In 9 cases, $Q(x, y, t)$ is x, y-DA but neither is holon.

• 2. O. Bernardi, M. Bousquet-Mélou, K. Raschel first proved that $Q(x, y, t)$ is x-,y-, or t-DA.

What about weighted models?

Weighted Models

- For 5 models with $|G| = \infty$, genus$(E) = 0$:



Theorem (Dreyfus-H.-Roques-Singer, 2020): For $t \in \mathbb{R} \backslash \overline{\mathbb{Q}}$

In all weighted cases, $Q(x, y, t)$ is not x, y-DA.

Weighted Models

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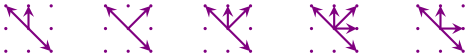

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Weighted Models

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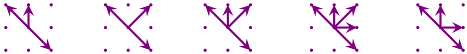


Theorem (Dreyfus-H.-Roques-Singer, 2020): For $t \in \mathbb{R} \backslash \overline{\mathbb{Q}}$

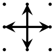
In all weighted cases, $Q(x, y, t)$ is not x, y-DA.

For weighted models with genus$(E) = 1$

some are and some are not

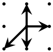
Examples (H.-Singer-2020)

Ex. 1 The weighted model

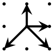


is always holonomic.

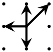
Ex.2 The nine unweighted differentially algebraic models with NSC on weights for DA



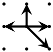
$w_{\mathrm{IIB.1}}$



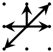
$w_{\mathrm{IIB.2}}$



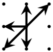
$w_{\mathrm{IIC.1}}$



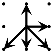
$w_{\mathrm{IIB.3}}$



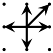
$w_{\mathrm{IIC.4}}$



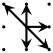
$w_{\mathrm{IIC.2}}$



$w_{\mathrm{IIB.6}}$



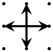
$w_{\mathrm{IIC.5}}$



$w_{\mathrm{IIB.7}}$

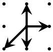
Examples (H.-Singer-2020)

Ex. 1 The weighted model



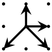
is always holonomic.

Ex.2 The nine unweighted differentially algebraic models with NSC on weights for DA



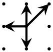
$w_{\mathrm{IIB.1}}$

All



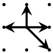
$w_{\mathrm{IIB.2}}$

All



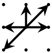
$w_{\mathrm{IIC.1}}$

All



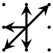
$w_{\mathrm{IIB.3}}$

All



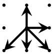
$w_{\mathrm{IIC.4}}$

$d_{-1,-1}d_{1,1} - d_{1,0}d_{-1,0} = 0$



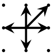
$w_{\mathrm{IIC.2}}$

$d_{0,1}d_{0,-1} - d_{1,1}d_{-1,-1} = 0$



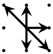
$w_{\mathrm{IIB.6}}$

All



$w_{\mathrm{IIC.5}}$

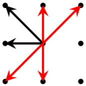
All



$w_{\mathrm{IIB.7}}$

$d_{-1,1}d_{1,-1} - d_{0,-1}d_{0,1} = 0$

Ex.3 An unweighted differentially transcendental model with NSC on weights for DA



$w_{\mathrm{IIB.6}}$

$$d_{-1,1}d_{0,1}^2 - d_{0,1}d_{-1,-1}d_{0,-1} + d_{1,1}d_{-1,-1}^2 = 0$$

- Generalities about Walks: Functional Equation, Curve, Group

- Theorems for Differential Algebraicity: Certificates, Decoupling

- Algorithms for Differential Algebraicity: Mordell-Weil Lattices, Néron-Tate Height

Generalities: Functional Equation of the Walk

Generating series: Fix $\mathcal{W}$ (and therefore $\mathcal{D}$)

$$Q(x,y,t) = \sum_{l,s,k} q(l,s,k) x^l y^s t^k$$

Step Inventory: $S(x,y) = \sum_{(i,j)} d_{i,j} x^i y^j$

Kernel polynomial: $K(x,y,t) = xy(1 - tS(x,y)) = xy - t\widetilde{S}(x,y)$ - *biquadratic*

Functional Equation: via an induction on the length

$$K(x,y,t)Q(x,y,t) = xy - K(x,0,t)Q(x,0,t) - K(0,y,t)Q(0,y,t) + K(0,0,t)Q(0,0,t).$$

Prop.: $Q(x,y,t)$ is x,y-DA $\Leftrightarrow$ $Q(x,0,t)$ is x-DA and $Q(0,y,t)$ is y-DA.

Fix $t_\alpha \in \mathbb{C}\backslash\overline{\mathbb{Q}}$. The kernel curve is the curve

$$E = \{([x_0 : x_1], [y_0 : y_1]) \mid x_1 y_1 K(\frac{x_0}{x_1}, \frac{y_0}{y_1}, t_\alpha) = 0\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

E is a general member of a family of curves $E_{[t_0:t_1]}$ parametrized by

$t = [t_0 : t_1] \in \mathbb{P}^1$ given by

$$x_1 y_1 t_1 K(\frac{x_0}{x_1}, \frac{y_0}{y_1}, \frac{t_0}{t_1}) = t_1 x_0 x_1 y_0 y_1 - t_0 \widetilde{S}([x_0 : x_1], [y_0 : y_1]) = 0$$

E_0 corresponds to the equation $x_0 x_1 y_0 y_1 = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$.
E_∞ corresponds to the equation $\widetilde{S}([x_0 : x_1], [y_0 : y_1]) = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$.
These two curves intersects in 8 *base points* and all the curves of the family pass through these points.

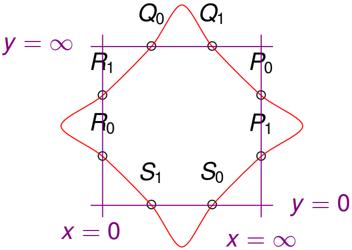
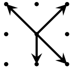
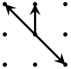


Figure: Position of the base points

Fact: K irreducible $\Rightarrow$ E has genus 0 or 1. This can be read in the set of directions.

Ex: 1) $\mathcal{D} =$  $E : xy - t(y^2 + x^2y^2 + x^2 + x) = 0 \;\; \Rightarrow g(E) = 1$

2) $\mathcal{D} =$  $E : xy - t(y^2 + xy^2 + x^2) = 0 \;\; \Rightarrow g(E) = 0$

for $t \in \mathbb{C} \backslash \overline{\mathbb{Q}}$

Parametrization

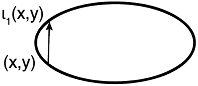
- If $g(E) = 0$ then $\phi : \mathbb{P}^1 \to E, z \mapsto (x(z), y(z))$ given by rational fractions;
- If $g(E) = 1$ then $\phi : \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \to E, z \mapsto (x(z), y(z))$ given by Weierstrass $\wp$-functions.

Group of the Walk

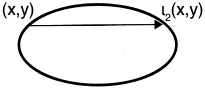
$$E = \overline{\{K(x,y,t_\alpha)=0\}} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

Since K is biquadratic in x and y, we define two involutions of E and an automorphism:

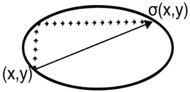
$$\iota_1(x,y) = (x, \frac{1}{y}\frac{\sum_i d_{i,-1}x^i}{\sum_i d_{i,1}x^i})$$



$$\iota_2(x,y) = (\frac{1}{x}\frac{\sum_j d_{-1,j}y^j}{\sum_j d_{1,j}y^j}, y)$$



$$\sigma = \iota_2 \circ \iota_1$$

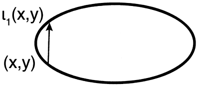


Group of the Walk

$$E = \overline{\{K(x, y, t_\alpha) = 0\}} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$$

Since K is biquadratic in x and y, we define two involutions of E and an automorphism:

$$\iota_1(x, y) = \left(x, \frac{1}{y} \frac{\sum_i d_{i,-1} x^i}{\sum_i d_{i,1} x^i}\right)$$



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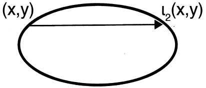

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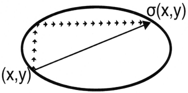


The Group of the Walk G is the group generated by ι_1, ι_2.

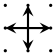
$$|G| < \infty \Leftrightarrow \exists n \text{ such that } \sigma^n = id.$$

Properties of the group

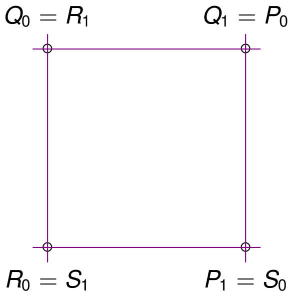
- G is finite iff $Q(x,y)$ is holonomic (A. Bostan, M. Bousquet-Mélou, T. Dreyfus, M. van Hoeij, M. Kauers, M. Mishna, K. Raschel)
- $g(E) = 0 \Rightarrow \sigma(z) = qz$ and q is not a root of unity (D-H-R-S).
- $g(E) = 1 \Rightarrow \exists \Omega_3 \in E$, s.t. $\sigma(\mathbf{Q}) = \mathbf{Q} \oplus \Omega_3$ Thus,

$$\sigma^n(Q) = Q \text{ for some } Q \leftrightarrow \sigma^n = id.$$

The weighted cross:



The automorphism σ has always order two.



since $\sigma^2(P_1) = P_1$. The generating series is holonomic

From now on, we will assume that E has genus 1 and G is infinite.

that is,
E is an elliptic curve and
$\sigma(Q) = Q \oplus \Omega_3$ for any $Q \in E$ and Ω_3 is non-torsion.
(on the universal cover $\sigma(z) = z + \omega_3$)

Theorems for Differential Algebraicity

Theorem: (Kurkova-Raschel, 2012 (unweighted) and Dreyfus-Raschel 2019 (weighted))
For $t_\alpha \in \mathbb{R} \setminus \overline{\mathbb{Q}}$, one construct $\phi : \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \to E, z \mapsto (x(z), y(z))$ and $\sigma(z) = z + \omega_3$.
The function $Q(0, y, t_\alpha)$ can be analytically continued to a meromorphic function F over $\mathbb{C}$ such that

$$F(z + \omega_3) - F(z) = b,$$

where $b = x(\iota_1(y) - y) \in \mathbb{C}(E)$ the function field of E.
Prop. (Dreyfus-H.-Roques-Singer, 2018)
By differential Galois theory (H.-Singer-20008), the following are equivalent:

1. $Q(0, y, t_\alpha)$ is y-DA.
2. There exist an integer $n \geq 0$, $c_i \in \mathbb{C}$, and $g \in \mathbb{C}(E)$ such that

$$\delta^n(b) + c_{n-1}\delta^{n-1}(b) + \ldots + c_1\delta(b) + c_0 b = \sigma(g) - g.$$

with $\delta = \frac{d}{dz}$.

Condition 2 holds if an only if the orbit residues for b are zero.

A rational fraction $b = \frac{d}{dz}(g)$ if and only if residues of order one of poles of b are zero.

Orbitresidues are discrete generalization.

Theorems for Algebraicity: Orbit Residues

Def. E elliptic curve, σ the addition by a non-torsion point, $K = \mathbb{C}(E)$

- $\{u_\mathbf{Q} \mid \mathbf{Q} \in E\}$ local param. are **coherent** if $u_{\sigma^{-1}(\mathbf{Q})} = \sigma(u_\mathbf{Q})$.
- For $b \in \mathbb{C}(E)$, $\mathbf{Q} \in E$, write

$$b = \frac{c_{\mathbf{Q},N}}{{u_\mathbf{Q}}^N} + \cdots + \frac{c_{\mathbf{Q},i}}{u_\mathbf{Q}^i} + \cdots + \frac{c_{\mathbf{Q},1}}{u_\mathbf{Q}} + f$$

with f regular at $\mathbf{Q}$. Then, the **i$^{\text{th}}$ orbit residue** of b at $\mathbf{Q}$ is

$$\mathrm{ores}^i_\mathbf{Q}(b) = \sum_{n\in\mathbb{Z}} c^i_{\sigma^n(\mathbf{Q})}.$$

Prop. *(D-H-R-S (2018))* The following are equivalent

- $Q(x, y, t)$ is x, y-DA.
- For all $i \in \mathbb{N}_{>0}$, $Q \in E$, $\mathrm{ores}^i_Q(b) = 0$.

To determine if $Q(x, y, t)$ is DA

find the orbits of the poles of $b = x(\iota_1(y) - y)$ and their orbit residues.

Theorems for Algebraicity: Orbit Residues

To prove DA, show: for all $i \in \mathbb{N}_{>0}, Q \in E$, $\text{ores}^i_Q(b) = \sum_{n\in\mathbb{Z}} c^i_{\sigma^n(\mathbf{Q})} = 0$

Ex. The unweighted model

The polar divisor of b is $(b)_\infty = \mathbf{M} + \mathbf{N} + \iota_1(\mathbf{N})$ where $\mathbf{M}, \mathbf{N} \in \mathbb{P}^1 \times \mathbb{P}^1$ and

- $\mathbf{M} = ([1:0],[0:1])$ and $\iota_1(\mathbf{M}) = \mathbf{M}$; residue $= \alpha \neq 0$.
- $\mathbf{N} = ([-1:1],[1:0])$; residue $= \beta \neq 0$
- $\iota_1(\mathbf{N}) = ([-1:1], *)$; residue $= \beta \neq 0$

Classical Residue Theorem $\Rightarrow \alpha + 2\beta = 0$. Thus,
orbit residues are zero $\Leftrightarrow$ all poles are in same orbit (in particular, $\mathbf{M} = \sigma^n(\mathbf{N})$).

Differential Algebraicity $\iff$ $\mathbf{M} = \sigma^n(\mathbf{N})$ for some n.

Theorems for Differential algebraicity: Weighted models

Prop.(H.-Singer (2020)) For $t_\alpha \in \mathbb{R} \setminus \overline{\mathbb{Q}}$, the following are equivalent

1. $Q(0, y, t_\alpha)$ is y-DA over $\mathbb{C}(y)$;
2. There exist an integer $n \geq 0$, $c_i \in \mathbb{C}$, and $g \in \mathbb{C}(E)$ such that

$$\mathcal{L}(b) := \delta^n(b) + c_{n-1}\delta^{n-1}(b) + \ldots + c_1\delta(b) + c_0 b = \sigma(g) - g.$$

   for $b = x(\iota_1(y) - y) \in \mathbb{C}(E)$.
3. There exists $g \in \mathbb{C}(E)$ such that

$$b = \sigma(g) - g.$$

   This g is called a certificate
4. There exist $f(x) \in \mathbb{C}(x)$ and $h(y) \in \mathbb{C}(y)$ s.t.

$$xy = f(x) + h(y)$$

   in $\mathbb{C}(E)$. The model $\mathcal{W}$ is decoupled. Bernardi, Bousquet-Mélou, Raschel (2017) show that 9 models of 51 unweighted models with $|G| = \infty$, 9 were decoupled and how one can find an explicit differential equation out of this decoupling.
5. Two precise poles $\mathbf{Q}, \mathbf{P}$ of b which are also *base points of the pencil* are such that

$$\sigma^n(\mathbf{Q}) = \mathbf{P}$$

   for some $n \in \mathbb{Z}$.

First application: differential transcendence criteria

Remember the position of the base points

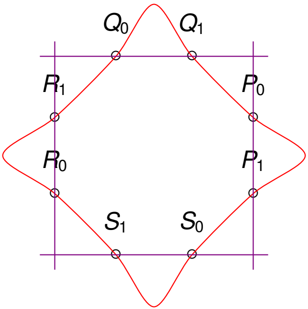
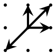


Figure: Position of the base points

Theorem (H.-Singer 2020): If one of the P_i's and one of the Q_j's is fixed by an involution then $Q(0, y, t_\alpha)$ is D-transcendental.
Example If its group G is infinite then a weighted model



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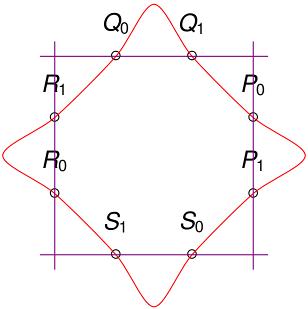

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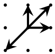


is D-transcendental.

Algorithms for differential algebraicity: Mordell-Weil Lattices, Néron-Tate Height

Fix a set of directions $\mathcal{D}$.
The question of the differential algebraicity of $Q(x, y, t_\alpha)$ is reduced to the following: given two points $\mathbf{Q}, \mathbf{P}$, is there some $n \in \mathbb{Z}$ such that

$$\mathbf{Q} = \sigma^n(\mathbf{P}).$$

Is there an algorithm to test this condition without fixing the weights on the set of directions?

Since E is defined over $k = \mathbb{Q}(t)$ and that E does not descend to $\mathbb{Q}$.

There is a height $\hat{h} : E(k) \mapsto \mathbb{Q}$ called Néron-Tate height such that

- If $\mathbf{Q}$ is a torsion point, then $\hat{h}(\mathbf{Q}) = 0$.
- If $\mathbf{Q} = n\mathbf{N}$, then $\hat{h}(\mathbf{Q}) = n^2 \hat{h}(\mathbf{N})$.
- $\hat{h}(\mathbf{N})$ is computable. *For the points we consider, this depends on the configuration of base points of the family $K(x, y, t) = 0$ and certain lines in a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$, not on the weights.*

Test $\mathbf{Q} = \sigma^n(\mathbf{P})$
On an elliptic curve we can select the point we call $\mathbf{O}$. Let $\mathbf{O} = \mathbf{P}$. We have

$$\sigma(\mathbf{P}) = \mathbf{P} \oplus \mathbf{N} = \mathbf{O} \oplus \mathbf{N} = \mathbf{N}$$

so $\sigma^n(\mathbf{P}) = n\mathbf{N}$. Therefore

$$\mathbf{Q} = \sigma^n(\mathbf{P}) \Leftrightarrow \mathbf{Q} = n\mathbf{N}.$$

So $\mathbf{Q} = \sigma^n(\mathbf{P}) \Rightarrow \hat{h}(\mathbf{Q}) = n^2\hat{h}(\mathbf{N})$.
This allows to find the potential n and the algebraic conditions on the weights by comparing $\mathbf{Q}$ and $\sigma^n(\mathbf{P})$.

Computing the heights without fixing the weights

$$K(x, y, t) = xy - txyS(x, y)$$
$$S(x, y) = \sum_{(i,j)} d_{i,j} x^i y^j$$

Let $\mathcal{S}$ be the Kodaira-Néron model of E. Then,

- $\mathcal{S}$ is a smooth projective rational surface with projection $\pi : \mathcal{S} \to \mathbb{P}^1$
- Almost all fibers are isomorphic to E; $\mathcal{S}_0$ is singular.
- Bijection between $\mathbb{Q}(t)$-points $\mathbf{P}$ of E and sections $\boldsymbol{\mathcal{P}} : \mathbb{P}^1 \to S$ over $\mathbb{Q}$.

Eight base points distinct $\Rightarrow \mathcal{S} : t_1 x_0 x_1 y_0 y_1 - t_0 \tilde{S}(x_0, x_1, y_0, y_1) = 0$
If not, $\mathcal{S}$ is a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at the eight points and $\mathcal{S}_0$ is obtained from $E_0 : x_0 x_1 y_0 y_1 = 0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ via the successive blowups.

More About the Néron-Tate Height

$$E \Rightarrow \pi : \mathcal{S} \to \mathbb{P}^1$$

$$\mathbf{P} \in E(\mathbb{Q}(t)) \Leftrightarrow \boldsymbol{\mathcal{P}} : \mathbb{P}^1 \to \mathcal{S}$$

$$\text{Def. } \hat{h}(\mathbf{P}) = \hat{h}(\boldsymbol{\mathcal{P}}) = 2\chi(\mathcal{S}) + 2(\boldsymbol{\mathcal{P}}.\boldsymbol{\mathcal{O}}) - \sum_{\nu \in R} contr_\nu(\boldsymbol{\mathcal{P}})$$

$$= 2 - \sum_{\nu \in R} contr_\nu(\boldsymbol{\mathcal{P}})(\text{ for the points } \mathbf{P} \text{ we are considering})$$

$$\sum_{\nu \in R} contr_\nu(\boldsymbol{\mathcal{P}}) = ????$$

What we know about $\pi : \mathcal{S} \to \mathbb{P}^1$:

- R = set of reducible fibers
  - Finite number of types *(Kodaira, Néron)*
  - One can explicitly compute the type of $\mathcal{S}_0$ by blowup. It depends only from the position of the base points.
- The type of a reducible fiber at ν determines a root lattice T_ν
  - *(Oguiso-Shioda)* finitely many possibilities for $\oplus_{\nu \in R} T_\nu \subset E_8$
  - Knowing T_0 allows to reduce the number of possibilities.
- Root Lattice $T_\nu \Rightarrow$ finite set of values for $contr_\nu(\boldsymbol{\mathcal{P}})$ *(Shioda)*

Working through an example

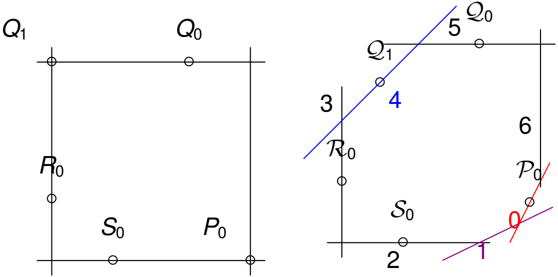
Ex. The weighted model 

It is DA

iff

$$Q_0 = \sigma^n(P_0) = n\sigma(P_0) = nS_0 \text{ for some } n$$

(choosing P_0 as the zero of E)



(a) $Q_1 = R_1$ and $P_0 = P_1 = S_1$ (b) The fiber above zero

Figure: Fiber above zero is an I_7

The steps of the algorithm are as follows

- $\mathcal{S}_0$ is an I_7 and $contr_0(S_0) = \frac{2(7-2)}{7}$ and $contr_0(Q_0) = \frac{5(7-5)}{7}$.
- Oguiso-Shioda list: $I_7 \Rightarrow A_6$
  Oguiso-Shioda list: $\Rightarrow$ at most 2 reducible fibers $\mathcal{S}_0, \mathcal{S}_\alpha$ and
  $\oplus_{\nu \in R} T_\nu = A_6$ or $A_6 \oplus A_1$.
- Shioda list: $T_\alpha = A_1 \Rightarrow contr_\nu(\mathcal{P}) \in \{0, 1/2\}$

If $\hat{h}(Q_0) = n^2 \hat{h}(S_0)$ for some $n \in \mathbb{Z}$ then

$$n^2(2 - \frac{10}{7} - \frac{\epsilon}{2}) = 2 - \frac{10}{7} - \frac{\epsilon'}{2}.$$

The only solutions are $n = \pm 1$.

- $n = 1$ is not possible since $S_0 \neq Q_0$
- $n = -1$ is equivalent to $d_{1,0}d_{-1,0} - d_{-1,1}d_{1,-1} = 0$

Open problems

- Combinatorial interpretation
- Problems of specializations of the variables in $Q(x, y, t)$
- Walks in an orthant: replace the kernel curve by a kernel surface.
- Walks with large steps: the kernel curve is no longer given by a biquadratic polynomial.