A tale of two analyticities

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I - A fairy tale about reality

II - Reality about a fairy tale

I - A fairy tale about reality

Some non-mathematical definitions

Reality

Nice familiar algebraic and analytic geometry over $\mathbb{C}$!

Fairy Tale

1 b: a story in which improbable events lead to a happy ending

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(we'll assume some big conjectures!)

The setup

General:

- $S/\overline{\mathbb{Q}}$ is an algebraic variety.
- $f : X \to S$ is a smooth proper family of algebraic varieties.
- $\tilde{S}$ is the universal cover of $S(\mathbb{C})$.

Example (The Legendre family)

- $S = \mathbb{P}^1 \backslash \{0, 1, \infty\}$
- X_λ is the elliptic curve defined by $y^2 = x(x-1)(x-\lambda)$, i.e.

$$X = V\left(ZY^2 - X(X-Z)(X-\lambda Z)\right) \subset \underbrace{S}_{[\lambda:1]} \times \underbrace{\mathbb{P}^2}_{[X:Y:Z]}$$

- $\tilde{S} \cong \mathbb{H}$, the upper half plane.

The period mapping

General:

- Fix $s_0 \in S(\mathbb{C})$, $0 \leq k \leq 2 \dim X_{s_0}$, and a trivialization

$$\mathbb{Q}^n \xrightarrow{\sim} H^k(X_{s_0}, \mathbb{Q}).$$

- Let $m_p = \dim H^{k-p}(X_{s_0}, \Omega^p)$
- Let $\mathcal{F}l$ be the variety classifying decreasing flags of subspaces in $\mathbb{Q}^n$ with $\dim F^p/F^{p+1} = m_p$.
- We obtain a *period map*

$$\pi : \tilde{S}(\mathbb{C}) \to \mathcal{F}l(\mathbb{C})$$

measuring the position of the Hodge filtration with respect to the continuation of the chosen trivialization.

The period mapping - example

Example (The Legendre family continued)

- On X_λ, we have the holomorphic differential form dx/y.
- Integration of dx/y induces an element

$$\int_\bullet \frac{dx}{y} \in H^1(X_\lambda(\mathbb{C}), \mathbb{C}) = \mathrm{Hom}(H_1(X_\lambda(\mathbb{C}), \mathbb{Z}), \mathbb{C}).$$

- If we fix a trivialization of $\mathbb{Q}^2 \to H^1(X_1(\mathbb{C}), \mathbb{Q})$ then we obtain by continuation, for any $\tau \in \tilde{S}$, a trivialization

$$\varphi_\tau : \mathbb{Q}^2 \xrightarrow{\sim} H^1(X_{\lambda(\tau)}, \mathbb{Q})$$

- $\mathcal{F}l = \mathbb{P}^1$, and

$$\pi(\tau) = \varphi_\tau^{-1}\left(\langle \int_\bullet \frac{dx}{y} \rangle\right) \subset \mathbb{C}^2.$$

- π *identifies* $\tilde{S} = \mathbb{H} = \mathbb{P}^1(\mathbb{C}) \backslash \mathbb{P}^1(\mathbb{R})$.

A classical question

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\pi} & \mathcal{F}l(\mathbb{C}) \\ \downarrow u & & \\ S(\mathbb{C}) & & \end{array}$$

Question (Informal)

Which ($\overline{\mathbb{Q}}-$)algebraic conditions on the Hodge filtration induce ($\overline{\mathbb{Q}}-$)algebraic conditions on S?

Question (Formal)

For which irreducible ($\overline{\mathbb{Q}}-$)algebraic subvarieties $Z \subset \mathcal{F}l_{\mathbb{C}}$ is the analytic subset $u(\pi^{-1}(Z))$ a ($\overline{\mathbb{Q}}-$)algebraic subvariety of S?

The question for the Legendre family

$$\begin{array}{ccc} \mathbb{H} & \xhookrightarrow{\ \pi\ } & \mathbb{P}^1(\mathbb{C}) \\ \Big\downarrow \lambda & & \\ \mathbb{C}\backslash\{0,1\} & & \end{array}$$

The question

$Z \subset \mathbb{P}^1_{\mathbb{C}}$ irreducible subvariety means

1. Z is a point
2. $Z = \mathbb{P}^1_{\mathbb{C}}$

Only interesting question: when are both τ and $\lambda(\tau)$ in $\overline{\mathbb{Q}}$?

Theorem (Schneider[1], 1937)

Both τ and $\lambda(\tau)$ are in $\overline{\mathbb{Q}}$ if and only if $K = \mathbb{Q}(\tau)$ is a quadratic imaginary field (if and only if $K \xrightarrow{\sim} End(X_\tau) \otimes \mathbb{Q}$.)

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[1] Schneider's result is usually formulated with j instead of λ, but equivalent!

What we know - I

$$
\begin{array}{ccc}
\tilde{S} & \xrightarrow{\pi} & \mathcal{F}l(\mathbb{C}) \\
\big\downarrow u & & \\
S(\mathbb{C}) & &
\end{array}
$$

The Hodge locus

- Suppose t is a weight zero tensor on $\mathbb{Q}^n = H^k(X_{s_0}, \mathbb{Q})$. The *Hodge locus* $\mathrm{Hdg}(t) \subseteq \mathcal{F}l$ is the locus where $t \in \mathrm{Fil}^0$.
- The *Hodge conjecture* predicts that $\pi^{-1}(\mathrm{Hdg}(t))$ is the locus of τ such that t, up to Tate twist, is represented by an algebraic cycle on $X^m_{u(\tau)}$, $m >> 0$
- Can also interpret using Mumford-Tate groups = "Galois group of a Hodge structure".

What we know - II

Theorem (Cattani-Deligne-Kaplan, 1995)

1. *For any t, $u(\pi^{-1}(\mathrm{Hdg}(t)))$ is a $\mathbb{C}$-algebraic subvariety of S.*
2. *(Weil) If the Hodge conjecture holds, then it is $\overline{\mathbb{Q}}$-algebraic.*

Moreover...

These should be the "only" algebraic conditions on the Hodge filtration that impose algebraic conditions on S (i.e. anything else should be explainable in terms of these).

What we know - III

Some closely related notions

1. Special (irreducible components...) and weakly special subvarieties (products with points...)
2. Bialgebraic subvarieties (careful when π_{dR} is not injective...)

Some more results

- $\mathbb{C}$-bialgebraic is equivalent to *weakly* special very generally (Klingler, 2017).
- In the abelian type Shimura case, $\overline{\mathbb{Q}}$-bialgebraic = special. (Ullmo-Yafaev, 2012 – the crucial case of points is a result of Cohen and Shiga-Wolfart generalizing Schneider's result).
- If special points are $\overline{\mathbb{Q}}$-algebraic then special subvarieties are $\overline{\mathbb{Q}}$-algebraic (Klingler, Otwinowska, Urbanik 2020).

What we know if we believe **everything**...

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\pi} & \mathcal{F}l(\mathbb{C}) \\ \downarrow u & & \\ S(\mathbb{C}) & & \end{array}$$

Hodge loci are algebraic on S

"Everything" includes the Hodge conjecture, thus we already know the Hodge loci cut out $\overline{\mathbb{Q}}$-algebraic conditions on S.

Theorem (conditional sideways $\overline{\mathbb{Q}}$-Ax-Lindemann; well-known?)

Suppose the Grothendieck period conjecture and the Hodge conjecture hold. If $Y \subset S$ is a smooth $\overline{\mathbb{Q}}$-algebraic subvariety and $\tilde{Y}$ is a connected component of $u^{-1}(Y(\mathbb{C}))$, then $\overline{\pi(\tilde{Y})}^{\overline{\mathbb{Q}}-\mathrm{Zar}} = \mathrm{Hdg}(t)$ for some t (t cuts out generic MT group).

Proof Sketch (still assuming **everything**)

1. Reduce to the Hodge-generic case:

$$
\begin{array}{ccl}
\tilde{Y} & \xrightarrow{\quad \pi \quad} & \underbrace{\mathcal{F}l_G(\mathbb{C})}_{\substack{G=\mathrm{Stab}(t) \\ \text{Generic MT group}}} = \mathrm{Hdg}(t) \subset \mathcal{F}l(\mathbb{C}) \\
\Big\downarrow u & & \\
Y(\mathbb{C}) & &
\end{array}
$$

2. Andre: There is a point $y \in Y(\overline{\mathbb{Q}})$ with $MT(\pi(\tilde{y})) = G$.
3. Grothendieck Period Conjecture $\Longrightarrow$ $\pi(\tilde{y})$ is a $\overline{\mathbb{Q}}$-generic point in $\mathcal{F}l_{MT(\pi(\tilde{y}))}(\mathbb{C})$.

Remark on Ullmo-Yafaev's bialgebraicity for Shimura varieties

Replace 2-3 with Deligne-Andre theorem to find enough monodromy to get a weakly special subvariety, then use weaker transcendence result (Wustholz) via Cohen/Shiga-Wolfart.

II - Reality about a fairy tale

Some non-mathematical definitions

Reality

We prove a result unconditionally.

Fairy Tale

1 a: a story (as for children) involving fantastic forces and beings (such as fairies, wizards, and goblins)
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(the world we work in is a bit more exotic!)

p-adic cohomology (Scholze, Bhatt-Morrow-Scholze)

Some fields

- $\breve{\mathbb{Q}}_p$ – like $\mathbb{Q}_p$ but start with $\overline{\mathbb{F}}_p$ instead of $\mathbb{F}_p$.
- $\overline{\breve{\mathbb{Q}}}_p$ - an algebraic closure. $\mathbb{C}_p$ is the p-adic completion of $\overline{\breve{\mathbb{Q}}}_p$.
- $B_{\mathrm{dR}} \cong \mathbb{C}_p((t))$ *abstractly*. $B_{\mathrm{dR}} \supset \overline{\mathbb{Q}_p}((t))$ *canonically*.

Cohomology

- $X/\overline{\breve{\mathbb{Q}}}_p$ smooth proper rigid analytic variety.

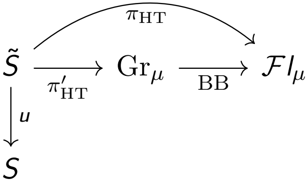
$$c_{\mathrm{dR}} : H^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(X) \otimes_{\overline{\breve{\mathbb{Q}}}_p} B_{\mathrm{dR}}$$

- $X/\mathbb{C}_p$ smooth proper rigid analytic variety.

$H^i(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} = \mathbb{M}_{\mathrm{dR}}[1/t]$, $\mathbb{M}_{\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}} \cong \mathbb{C}_p[[t]]} \mathbb{C}_p = H^i_{\mathrm{dR}}(X)$.

Lattices give rise to trace **Hodge-Tate filtration** on $H^i(X, \mathbb{Q}_p) \otimes \mathbb{C}_p$ and **Hodge filtration** on $H^i_{\mathrm{dR}}(X)$.

The setup



1. $S/\breve{\mathbb{Q}}_p$ a smooth rigid analytic variety.
2. X/S is a smooth proper family of rigid analytic varieties.
3. $\tilde{S}/S$ any profinite étale *diamond* trivializing cover for the local system $H^i(X_s, \mathbb{Q}_p)$.

5a. $\pi_{\mathrm{HT}} : \tilde{S} \to \mathcal{F}l_\mu$ measures the position of the *Hodge-Tate filtration* with respect to the trivialization.

5b. Upgrade: $\pi'_{\mathrm{HT}} : \tilde{S} \to \mathrm{Gr}_\mu$ (moduli space of lattices).

The question

Some remarks

1. $\mathcal{F}l_\mu$ and S are rigid analytic varieties over $\overline{\breve{\mathbb{Q}}}_p$.
2. $\tilde{S}$ typically departs from this world (lives somewhere between rigid analytic varieties and perfectoid spaces).
3. Gr_μ also departs from this world, but rigid analytic subvarieties make sense still. In fact, they are just the rigid analytic subvarieties of $\mathcal{F}l_\mu$ satisfying Griffiths transversality.
4. In situations related to abelian varieties (or p-divisible groups), μ is *miniscule* so $\mathcal{F}l_\mu = \mathrm{Gr}_\mu$.

Question

Which rigid analytic conditions on the Hodge-Tate filtration (or lattice) induce rigid analytic conditions on S?

Example (The Legendre family)

For the Legendre family we have $\mathrm{Gr}_\mu = \mathcal{F}l_\mu = \mathbb{P}^1$.

$$\begin{array}{ccc} \tilde{S} & \xrightarrow[\pi'_{\mathrm{HT}}=\pi_{\mathrm{HT}}]{} & \mathbb{P}^1 \\ \downarrow{\scriptstyle \lambda} & & \\ \mathbb{P}^1\backslash\{0,1,\infty\} & & \end{array}$$

$$\text{For } \tau \in \mathbb{P}^1(\mathbb{C}_p), \lambda(\pi_{\mathrm{HT}}^{-1}(\tau)) = \begin{cases} \text{Profinite set} & \text{if } \tau \notin \mathbb{P}^1(\mathbb{Q}_p) \\ \text{Dense interior} & \text{if } \tau \in \mathbb{P}^1(\mathbb{Q}_p). \end{cases}$$

Theorem (H., 2018 - local p-adic Schneider)

For $x \in \tilde{S}(\mathbb{C}_p)$, $\tau := \pi_{\mathrm{HT}}(x)$,

$$\tau \in \mathbb{P}^1(\overline{\breve{\mathbb{Q}}}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p) \text{ and } \lambda(x) \in \overline{\breve{\mathbb{Q}}}_p \Leftrightarrow [\mathbb{Q}_p(\tau) : \mathbb{Q}_p] = 2,$$

$(\Leftrightarrow \mathrm{End}(X_\lambda[p^\infty]) \otimes \mathbb{Q}_p = \mathbb{Q}_p(\tau))$.

How to generalize?

Work locally

1. Want to work only with S where the connection on de Rham cohomology is flat (+ a bit more).
2. Example: in the reduction disk of a point in the Legendre family (ordinary $\leftrightarrow \mathbb{P}^1(\mathbb{Q}_p)$, supersingular $\leftrightarrow \mathbb{P}^1 \backslash \mathbb{P}^1(\mathbb{C}_p)$).
3. Have this behavor locally on reduction disks for a smooth proper formal model.
4. **Compare analyticity of period domain for Hodge filtrations to analyticity of period domain for Hodge-Tate filtrations.**

The universal case for p-divisible formal groups

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\pi_{\mathrm{HT}}} & \mathcal{F}l_{\mathrm{HT}} \\ \downarrow{\scriptstyle \pi_{\mathrm{dR}}} & & \\ \mathcal{F}l_{\mathrm{Hodge}}^{b-\mathrm{adm}} & & \end{array}$$

- d=dimension, n=height.
- $\mathcal{F}l_{\mathrm{Hodge}}$ and $\mathcal{F}l_{\mathrm{HT}}$ both (classical) Grassmannians for d-dimensional subspaces of n-dimensional vector space.
- b encodes Newton polygon, b-admissible locus is open.
- Image of π_{HT} is locally closed, open if b semistable/isoclinic.

Theorem (H., Klevdal - Rough version)

If S is a smooth rigid analytic variety over a finite extension of $\breve{\mathbb{Q}}_p$, $f : S \to \mathcal{F}l_{\mathrm{Hodge}}^{b-\mathrm{adm}}$, and $\tilde{S}$ is a connected component of $\pi_{\mathrm{dR}}^{-1}(S)$, then $\pi_{\mathrm{HT}}(\tilde{S}) \subset \mathcal{F}l_G$, G the generic MT group, and any rigid analytic subset of $\mathcal{F}l_G$ containing $\pi_{\mathrm{HT}}(\tilde{S})$ has non-empty interior.

In general

- $\mathcal{M} = \mathcal{M}_{G,\mu,b}$ moduli of mixed characteristic local shtuka (allow $G/\mathbb{Q}_p$ arbitrary linear algebraic! No reason for local MT groups to be reductive here, even in case above)
- The flag varieties are replaced by diamond affine Grassmannians
- Maps from smooth rigid analytic subvarieties still make sense! (They correspond exactly to maps to the flag variety satisfying Griffiths transversality).
- For b-basic, can swap π_{HT} and π_{dR} – get bialgebraicity/ "Ax-Lindemann" type results for this (most important) case.

Proof sketch

1. Reduce to Hodge generic case.
2. Find $\overline{\breve{\mathbb{Q}}}_p$-point in Hodge generic locus (we have to do some work in general here, though in some "structurally polarized" cases the Hodge generic locus is just a dense open!)
3. Observe that Fontaine's crystalline comparison theorem is a strong version of a local Grothendieck Period Conjecture.

$$c_{\mathrm{dR}} : H^i(X_s, \mathbb{Q}_p) \otimes B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(X_s) \otimes B_{\mathrm{dR}}$$

For $\sigma \in \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/K)$,

$$\sigma(c_{\mathrm{dR}}) = c_{\mathrm{dR}} \circ \rho(\sigma)^{-1}$$

for $\rho : \mathrm{Gal}(\overline{\breve{\mathbb{Q}}}_p/K) \to G(\mathbb{Q}_p)$ with open image where G is the Mumford-Tate group of the local p-adic Hodge structure (this Mumford-Tate group makes sense not just at $\overline{\breve{\mathbb{Q}}}_p$-points!).

Thanks for coming!

- Questions? (if time)
- Contact: sean.howe@utah.edu
- Preprint available soon we hope!