Kudla-Rapoport special cycles and More AFL conjectures

Kudla Fest

July 6^{th} 2021

Plan

I) Arithmetic GGP for Bessel models

II) RZ spaces for non-red group

III) Connection to KR cycles

IV) AFL conjectures for "Bessel cycles"

I) AGGP

G reductive $\mathbb{Q}$

(H, G) spherical G/H

Example $G = U(m) \times U(m+1)$

$H = U(m)$

$$\langle Sh_H \pi, Sh_H \pi \rangle_{Sh_G} = L'(\pi, \tfrac{1}{2})$$

H red $\Longleftrightarrow$ G/H affine

Today: H non-red.

Photo courtesy: Jeff Mozzochi.

Bessel subgroup

Fix $n = m+1+2r$ $\qquad m \geq 1, \quad r \geq 0$

F/F_0 : quadratic, $\quad V$ = Hermitian space

$\dim_F V = n$

$G(V) = U(V)$

Filtration:

$\perp$

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r \subseteq V_{r+1} \cdots \subseteq V_{2r+1} = V$$

$W^\# := V_{r+1}/V_r \qquad \dim W^\# = m+1$

$u \in W^\# \qquad W = \langle u \rangle^\perp$

$(u, u) \neq 0$

parabolic subgroup & Bessel subgroup

$$\begin{tikzcd}
H \arrow[r, hook] \arrow[d] & P \arrow[r] \arrow[d] & G(V) \\
G(W) \arrow[r, hook] & M = G(W^\#) \times \prod_{i=1}^{r} GL_{1,F} &
\end{tikzcd}$$

Bessel subgp

$$H \longrightarrow G = G(W) \times G(V)$$

$r = 0 \qquad H \cong U(W)$

$V = W^\#$

(Integral) RZ datum (unramified, EL/PEL)

$F / \mathbb{Q}_p$: finite unramified

V : F-vect space

$(\ ,\ ) : V \times V \longrightarrow \mathbb{Q}_p$ (PEL case)

$G := GL_F(V)$ $\mathbb{Q}_p$-alg gp

$\mu : \mathbb{G}_m \longrightarrow G_{\overline{\mathbb{Q}}_p}$

$[b] \in B(G) =$ σ-conj classes in $G(\breve{\mathbb{Q}}_p)$

$\left(\begin{array}{l} \breve{\mathbb{Q}}_p = W(\mathbb{F})[\frac{1}{p}] \\ \mathbb{F} = \overline{\mathbb{F}}_p \end{array} \right)$

$\Lambda \subseteq V$ $\mathcal{O}_F$-lattice (of full rank)

self dual (PEL case)

Remark:

① $(V_{\breve{\mathbb{Q}}_p}, b\sigma)$: isocrystal.

will assume: $[b]$ <u>basic</u>.

② from μ, $V_{\overline{\mathbb{Q}}_p} = V^0 \oplus V^1$.

$V_{i,\overline{\mathbb{Q}}_p} = V_i^0 \oplus V_i^1$

Filtered RZ datum

$$\left\{ \begin{array}{l} 0 = V_0 \subseteq \cdots \subseteq V_r \subseteq V \\ \mu \longrightarrow P_{\overline{\mathbb{Q}}_p} \longrightarrow G_{\overline{\mathbb{Q}}_p} \\ 0 = V_0 \subseteq \cdots \subseteq V_r \subseteq V_{r+1} \subseteq \cdots \subseteq V_{2r+1} = V \end{array} \right.$$

RZ formal moduli space

$\mathrm{Nilp} = $ Schemes / $\mathrm{Spec}\, \breve{\mathbb{Z}}_p$
p locally nilpotent

(X, ι) (PEL case: (X, ι, λ))

$$\begin{cases} X : & p\text{-div gp} / S \in \mathrm{Nilp} \\ \iota : & O_F \longrightarrow \mathrm{End}(X) \\ (\lambda : & X \longrightarrow X^\vee \quad \text{prin polarization.}) \end{cases}$$

s.t. <u>Kottwitz sign</u>: $\mathrm{Char}(a \mid \mathrm{Lie}\, X) = \mathrm{Char}(a \mid V_0)$
$a \in O_F$

Also fix a <u>framing object</u>

$(\mathbb{X}, \iota.)$ $\quad (\mathbb{X}, \iota, \lambda)$ over $\mathrm{Spec}\, \mathbb{F}$

Moduli functor of Rapoport-Zink.

$$\mathcal{N} : \mathrm{Nilp} \longrightarrow \mathrm{Sets}$$
$$S \longmapsto \{ (X, \iota, \rho_X) \}$$

$\rho_X : X \times_S \overline{S} \longrightarrow \mathbb{X} \times_{\mathbb{F}} \overline{S}$: quasi-isogeny
framing
$\overline{S} = S \times_{\breve{\mathbb{Z}}_p} \mathbb{F}$

<u>Theorem</u> (RZ)

$\mathcal{N}$: formally smooth over $\mathrm{Spf}\, \breve{\mathbb{Z}}_p$
locally noetherian formal scheme.

Example (EL case)

$F = \mathbb{Q}_p.$ $\qquad G \cong GL_n . \mathbb{Q}_p$

$\begin{cases} X: & p\text{-div gp} \ / S \in \mathrm{Nil}_p \\ \iota: & \mathbb{Z}_p \longrightarrow \mathrm{End}(X) \end{cases}$

(Kottwitz sign. $\Longleftrightarrow$ dimension of X)
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad \| $
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \dim V_0 = d$

framing object:

$\mathbb{X}$: p-div gp over $\mathbb{F}$
of height $= n$, $\dim = d$

Example

(1) $d = 1$. Lubin-Tate space $\circlearrowleft J = D_{1/n}^{\times}$

(2) $d = 3$. $n = 6$ (slope $= 1/2$)

$\mathbb{X} = \mathbb{E} \times \mathbb{E} \times \mathbb{E}.$

$\mathbb{E}$: "supersingular" $\dim = 1$
height $= 2$

$$\begin{array}{c} \mathcal{N} \circlearrowleft J = GL_3(B) \\ \Big\downarrow \\ \breve{\mathbb{Z}}_p \end{array}$$

rel dim $= 3 \times 3$

$B =$ quaternion nonsplit

Example : unitary RZ space

$F_0 \ (= \mathbb{Q}_p)$, F = unram quad of F_0

$V : F/F_0$ - Hermitian space

$\Lambda \subseteq V$: self dual lattice

$\mu \rightsquigarrow$ sign of F action on $V \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$

(r, s)

(X, ι, λ) : unitary p-div gp

$\iota : O_F \longrightarrow \mathrm{End}(X)$

$\lambda : X \longrightarrow X^\vee$

Kottwitz condition : $\mathrm{Char}(a \mid \mathrm{Lie}\, X) = (T-a)^r (T-\bar{a})^s \in O_S[T]$

$$\mathcal{N}_n : \mathrm{Nilp} \longrightarrow \mathrm{Sets}$$

Height = 0

$$S \longmapsto (X, \iota, \lambda, \rho)$$

$\mathcal{N}_n \longrightarrow \mathrm{Spf}\, O_{\breve{F}}$ rel dim $= rs$

Will assume : $(r, s) = (n-1, 1)$

Special case : $n = 1$

$$\mathcal{N}_1 = \mathrm{Spf}\, O_{\breve{F}},$$

$(\mathcal{E}, \iota, \lambda)$ canonical lifting

RZ space for "non-red" group

$P = GL_E(V, \mathrm{Fil}) \subseteq G$ (EL / PEL case)

"filtered" RZ datum (unramified)

$(\mathbb{X}_\bullet, \iota, \lambda)$ $\qquad X_\bullet = 0 = X_0 \subseteq X_1 \subseteq \cdots X_r = X$

$$\begin{tikzcd} \mathcal{N}_P \arrow[r] \arrow[d, "\pi"] & \mathcal{N}_G \\ \mathcal{N}_M & \end{tikzcd}$$

Theorem: π smooth.
(dim computable.)

Example

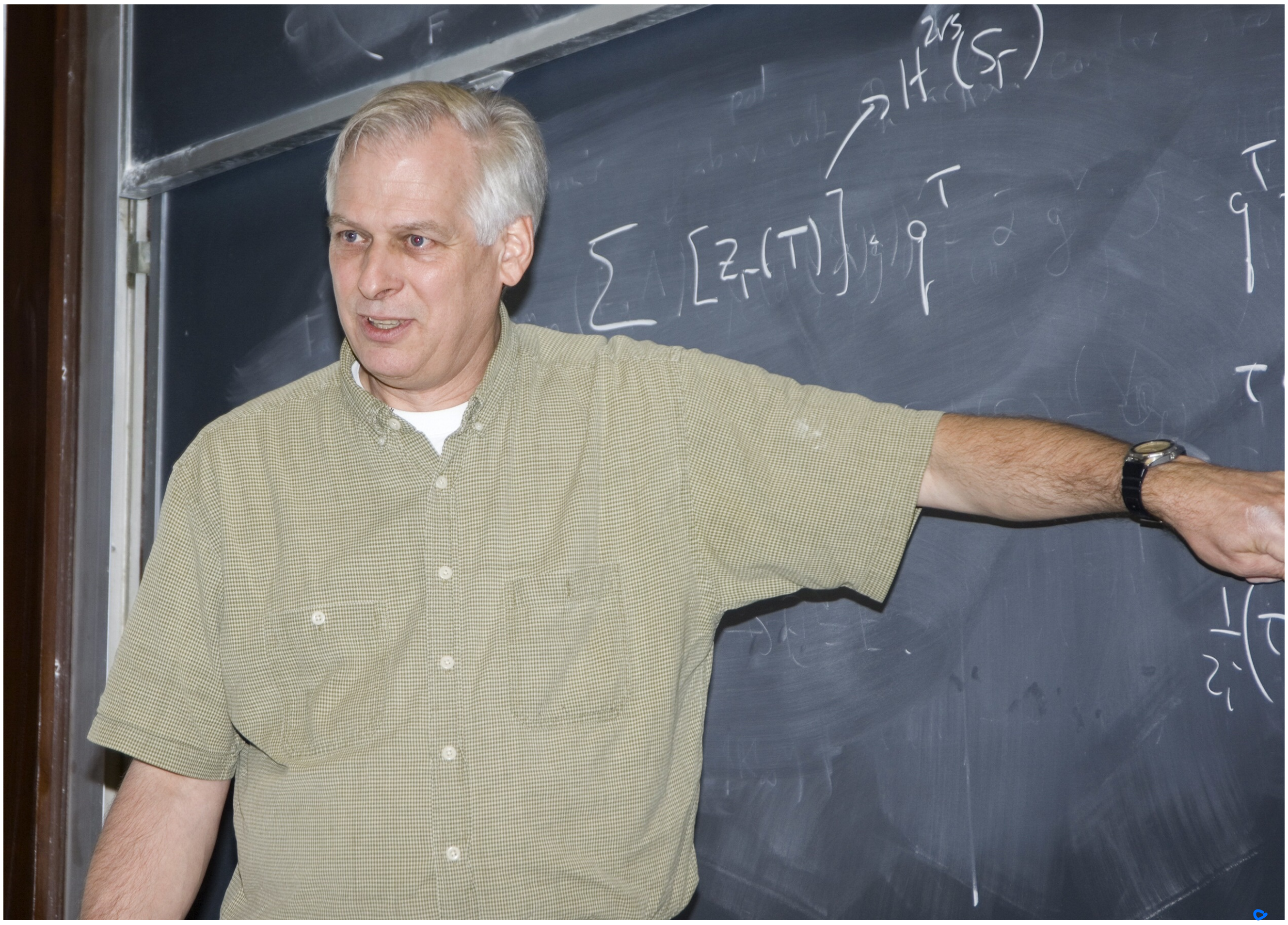
(2) $P = \left\{ \begin{pmatrix} \square_2 & * & * \\ & \square_2 & * \\ & & \square_2 \end{pmatrix} \right\} \subseteq G = GL_6 . \mathbb{Q}_p$

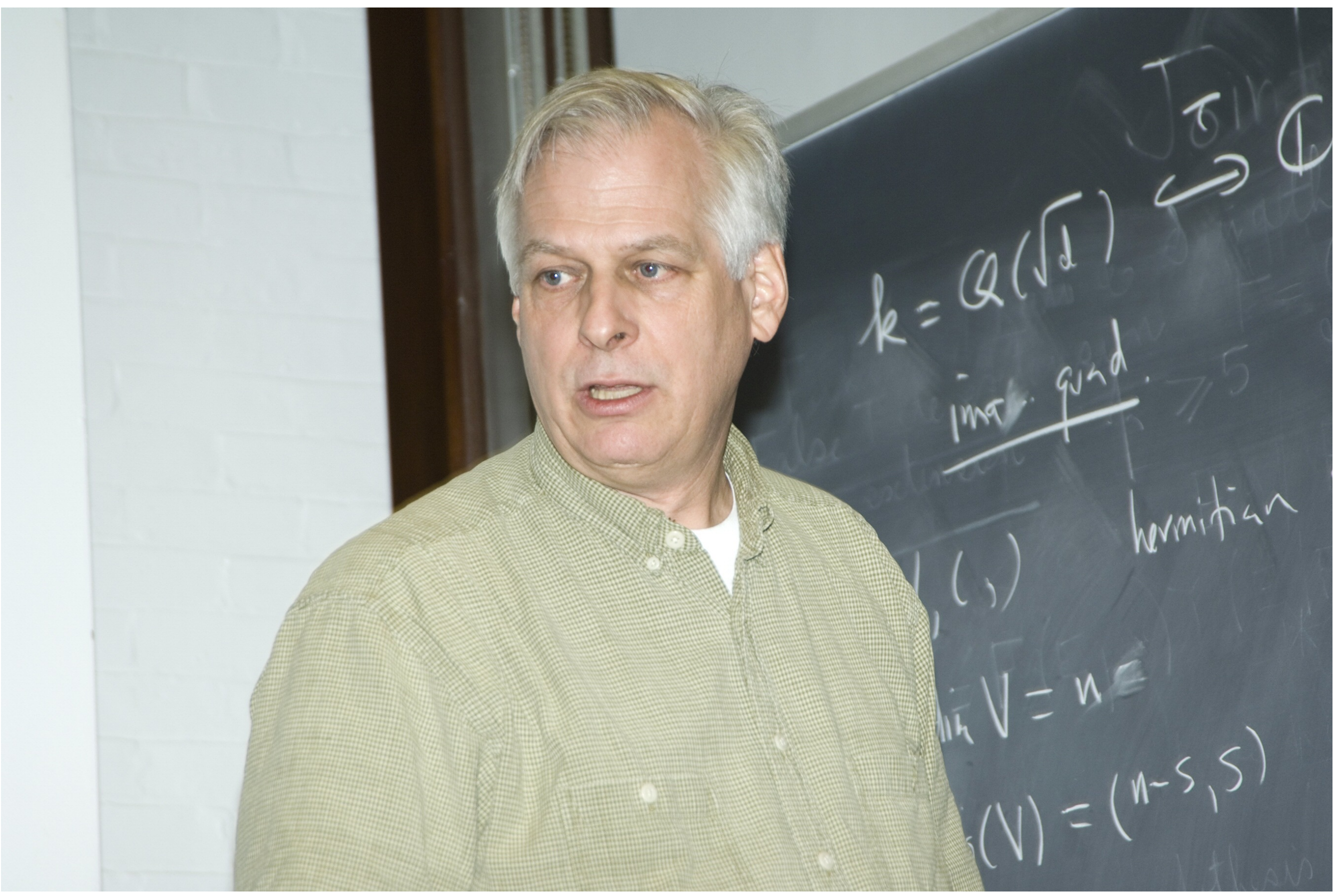
slope $1/2 = 3/6$

$$\begin{tikzcd} \mathcal{N}_P \arrow[r, "\text{loc closed}"] \arrow[d, "\text{rel dim}=3"'] & \mathcal{N}_G \\ LT \times_{\breve{\mathbb{Z}}_p} LT \times_{\breve{\mathbb{Z}}_p} LT & \end{tikzcd}$$

Kudla-Rapoport special cycles

Photo courtesy: Jeff Mozzochi





Global cycles

$$\mathcal{Z}(m) = \{ A_0 \xrightarrow{\varphi} A \}$$
$$\langle \varphi, \varphi \rangle = m \geq 0$$

$$\begin{tikzcd} & \mathcal{Z}(m) \arrow[dl] \arrow[dr] & \\ \mathcal{M}_1 & & \mathcal{M}_n \end{tikzcd}$$

$(A_0, \iota_0, \lambda_0)$. $\quad (A, \iota, \lambda)$

(can then form "Theta series":

$$\sum_{m \geq 0} [\mathcal{Z}(m)] q^m \in CH(\mathcal{M}) \otimes \mathbb{Z}[q] \;)$$

Local cycles

KR Hermitian space of "special homo"

$$u \in \mathbb{V}_n = \mathrm{Hom}_{O_F}(\mathbb{E}, \mathbb{X}_n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

$\boxed{\langle u . u \rangle \neq 0}$

$$\mathcal{Z}(u) = \left\{ \begin{tikzcd} \mathcal{E} \arrow[r, "\tilde{u}"] & X \\ \mathbb{E} \arrow[u, dashed, "\times S"', "\rho_0"] \arrow[r, "u"] & \mathbb{X}_n \arrow[u, dashed, "\times S"', "\rho_X"] \end{tikzcd} \right\}$$

$$\begin{tikzcd} & \mathcal{Z}(u) \arrow[dl] \arrow[dr] & \\ \mathcal{N}_1 & & \mathcal{N}_n \end{tikzcd}$$

$(\mathcal{E}, \iota_0, \lambda_0, \rho_0) \quad (X, \iota, \lambda, \rho_X)$

O_F - Lattice $\mathcal{L} \subseteq \mathbb{V} \rightsquigarrow \mathcal{Z}(\mathcal{L}) = \bigcap_{i=1} \mathcal{Z}(u_i)$

$\mathcal{L} = \langle u_1, \dots, u_r \rangle$

Fact: (a) $\mathcal{Z}(u)$, $u \neq 0$, relative Cartier div.

(b) $\operatorname{rank}(\mathcal{L}) = n \Rightarrow \mathcal{Z}(\mathcal{L})$ proper scheme.

Defn: $\mathcal{Z}(\mathcal{L})^\dagger$ = formally smooth locus of $\mathcal{Z}(\mathcal{L}) \to \operatorname{Spf} \breve{O}_F$

Theorem: (P as in Bessel subgp)

$$\mathcal{N}_P \xrightarrow{\sim} \bigsqcup_{\mathcal{L} \subseteq \mathbb{V}_r} \mathcal{Z}(\mathcal{L})^\dagger$$

O_F-lattices of $\mathrm{rk} = r$

totally isotropic

Remark Analogy

$$\begin{tikzcd}
\mathcal{Z}(\mathcal{L}) \arrow[r, leftrightarrow] \arrow[d, dashed, leftrightarrow] & \mathcal{Z}(\mathcal{L})^\dagger \arrow[d, dashed, leftrightarrow] \\
\text{Laumon compactification} \arrow[r, leftrightarrow] & \mathrm{Bun}_P
\end{tikzcd}$$

$P \subseteq GL_n$

Bessel cycle

(generalized arith. diagonal cycle)

$$\begin{tikzcd} \mathcal{N}_H \arrow[r] \arrow[d] & \mathcal{N}_P \arrow[r, "\text{loc. closed, } \mathrm{codim}=r"] \arrow[d, "\text{smooth, rel dim} = r"] & \mathcal{N}_n \\ \mathcal{N}_m \arrow[r] & \mathcal{N}_M & \end{tikzcd}$$

$$\mathcal{N}_H \longrightarrow \mathcal{N}_{n,m} = \mathcal{N}_n \times_{\mathrm{Spf}\, O_{\breve{F}}} \mathcal{N}_m$$

$$\mathrm{Int}(g) := \langle \mathcal{N}_H ,\ g\mathcal{N}_H \rangle_{\mathcal{N}_{n,m}}$$

$$g \in G(\mathbb{V}) \times G(\mathbb{W})$$

a) Add "weight" to $\mathcal{N}_H$

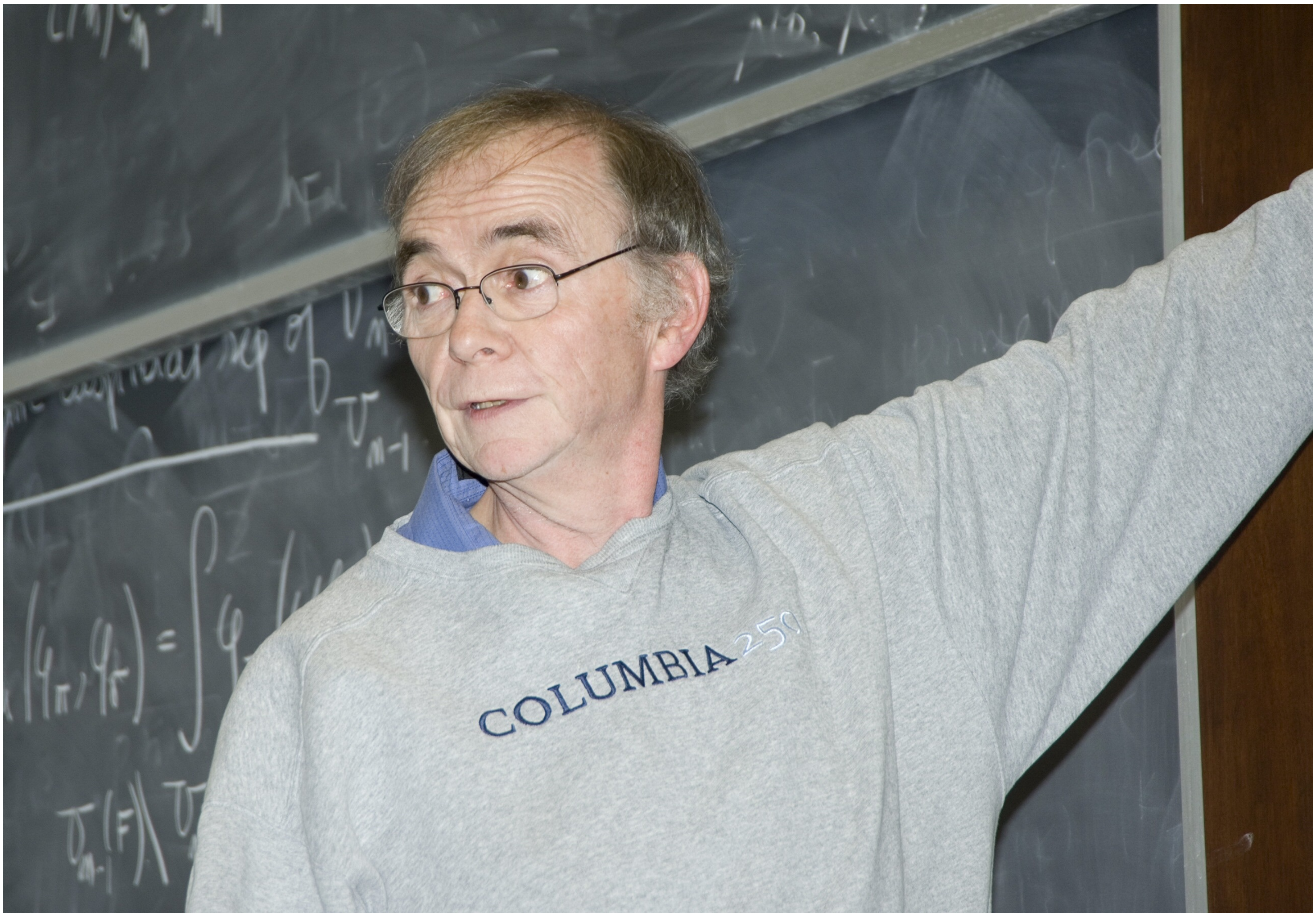
$$\boxed{\pi_0(\mathcal{N}_H) = ?}$$

b) compactification?

AFL conjecture for Bessel cycles

Relative Trace formula $\left\{ \begin{array}{l} \text{Jacquet-Rallis} \\ \text{Liu Yifeng} \end{array} \right.$

(Analytic / L-function side)



$$H \circlearrowright S_n \overset{\text{def}}{=} \{ \gamma \in GL_n F \mid \gamma \bar{\gamma} = 1_n \} \xrightarrow{\sim} GL_n F / GL_n F_0 :$$

$\exists$ orbital <u>integral</u> (J-R, Liu)

$$\mathrm{orb}(\gamma, \varphi) := \int_{\mathbb{H}(F_0)} \varphi(h^{-1}\gamma h)\, \eta(h)\, |\det(h)|^s \, dh$$

$(r = 0$ case $\quad \mathbb{H} = GL_{n-1, F_0}\ )$

<u>Conjecture</u> (AFL)

$$\pm\, \mathrm{Int}(g) \log p = \frac{d}{ds}\Big|_{s=0} \mathrm{orb}(\gamma, \mathbb{1}_{S_n(\mathcal{O}_{F_0})})$$

for all $g \longleftrightarrow \gamma \in S_n(F_0)_{\mathrm{reg}}$

<u>Remark</u> More examples: G/H <u>non affine</u>

a) Ginzburg–Rallis:

$$\begin{array}{ccccc} H & \longrightarrow & P & \longrightarrow & G = GL_6 \\ \downarrow & & \downarrow & & \\ GL_2 & \xrightarrow{\Delta} & GL_2^3 & & \end{array}$$

L-function: $\Lambda^3\, GL_6$

b) Exceptional groups

$$H \cong GL_2 \ltimes N \hookrightarrow E_7.$$

(Ginzburg, Wan-Zhang)

L-function: degree $= 56$.

dim of "Shimura variety" $= 27$.