The Fricke-Macbeath curve and triple product L-functions

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Steve and I were in the same undergraduate class at Harvard.

We took algebra from Richard Brauer.

We both participated in the summer school on automorphic forms and L-functions in Corvallis. Steve attended all the lectures...

We wrote one joint paper:

Heights and the central critical values of triple product L-functions. Compositio Math. **81** (1992).

The first part of this talk is joint work with Noam Elkies.

We were guided by a beautiful talk on the Fricke-Macbeath curve by Jaap Top.

The material on triple product L-functions was motivated by a surprising result in a recent paper of Dean Bisogno, Wanlin Li, Daniel Litt, and Padma Srinivasan.

Some important progress was made by Congling Qiu and Wei Zhang soon after the talk.

I want to thank all of them.

Hurwitz curves

Let X be a curve of genus $g \geq 2$ over $\mathbb{C}$ and let $G = \mathrm{Aut}(X)$.

The group G is finite and $\#G \leq 84(g-1)$.

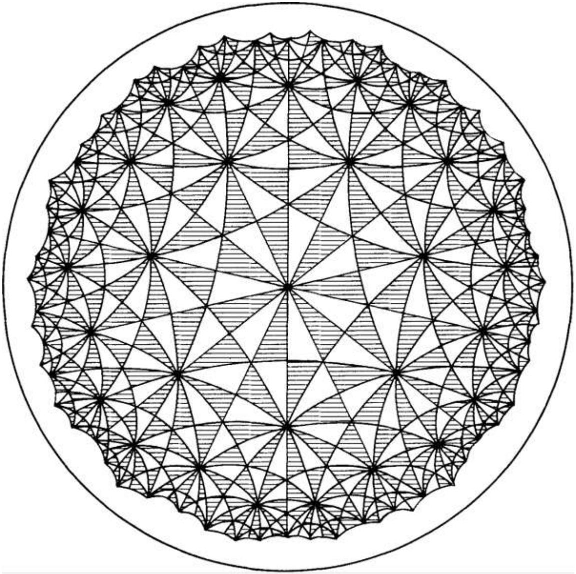
A. Hurwitz, "Über algebraische Gebilde mit Eindeutigen Transformationen in sich". Math. Annalen, **41** (1893).

If equality holds, then $X/G \cong \mathbb{P}^1$ and the covering $X \to \mathbb{P}^1$ is ramified at three points, with inertia subgroups of order $2, 3, 7$.

The finite group G is a quotient of the hyperbolic triangle group

$$\Delta(2,3,7) = \{x, y, z : x^2 = y^3 = z^7 = xyz = 1\},$$

which is a discrete subgroup of $\mathrm{Aut}(\mathfrak{H}) = \mathrm{SL}_2(\mathbb{R})/\langle \pm 1 \rangle$.



1879, F. Klein, Math. Annalen **14**

Let X be an H-curve, let $G = \mathrm{Aut}(X)$ and let Γ be the kernel of the surjective homomorphism $\Delta(2,3,7) \to G$.

We have a uniformization of the G-covering map

$$X(\mathbb{C}) = \Gamma\backslash\mathfrak{H} \to \mathbb{P}^1(\mathbb{C}) = \Delta(2,3,7)\backslash\mathfrak{H}.$$

The group G acts linearly on $H^0(X, \Omega)$.

If V is a non-trivial irreducible representation of G and V^* is its dual, then the Lefschetz fixed point formula implies

$$\mathrm{mult}(V) + \mathrm{mult}(V^*) = \dim V - \dim V^{\langle 2\rangle} - \dim V^{\langle 3\rangle} - \dim V^{\langle 7\rangle}.$$

The first few genera of H-curves are

$$g = 3, 7, 14, 17, 118, \ldots$$

The unique H-curve X of genus 3 is the Klein quartic

$$x^3y + y^3z + z^3x = 0$$

with automorphism group isomorphic to $\mathrm{PSL}_2(7)$.

The unique H-curve X of genus 7 is was studied by R. Fricke in 1899, as a quotient of the upper half plane.

It has automorphism group isomorphic to $\mathrm{PGL}_2(8)$.

The representation $V = H^0(X, \Omega)$ is irreducible and in the discrete series, associated to the two cubic characters of the non-split torus.

A.M. Macbeath found an algebraic model for X in 1965.

A Borel subgroup $B = U.T = 2^3.7$ of $\mathrm{PGL}_2(8)$ acts irreducibly on $H^0(X, \Omega)$.

The unipotent subgroup acts by the seven non-trivial characters $\chi : U \to \langle \pm 1 \rangle$, and these eigenspaces are permuted transitively by the torus.

If U_χ be the kernel of the character χ, the quotient $E_\chi = X/U_\chi$ is an elliptic curve.

The seven elliptic curves $E = E_\chi$ are all isomorphic, and

$$j(E) = 1792.$$

The Jacobian $J(X)$ is isogenous to the product E^7.

Since E does not have complex multiplication and $J(X) \sim E^7$ the group

$$L = \mathrm{Hom}(J(X), E)$$

is free abelian of rank 7.

This is the Mordell-Weil lattice

$$L = \mathrm{Mor}(X, E)/E(\mathbb{C}) = E(\mathbb{C}(X))/E(\mathbb{C}).$$

The degree of a morphism $\phi : X \to E$ defines a positive definite quadratic form on L.

The lattice L is isometric to $2(E_7^\vee) \subset E_7$ and has theta function

$$\theta_L = 1 + 56q^3 + 126q^4 + 576q^7 + 756q^8 + 1512q^{11} + ...$$

G. Shimura, "Construction of class fields and zeta functions of algebraic curves." Annals of Math. **85** (1967).

Let $k = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ be the cubic field of discriminant 7^2.

The ring of integers $A = \mathbb{Z}[\zeta_7 + \zeta_7^{-1}] \subset k$ has strict class number one: every ideal has a totally positive generator.

The field k has three real embeddings v, w, w'. Let B be the quaternion algebra over k which is split at v and all finite places and is ramified at w and w'.

The maximal A orders $R \subset B$ are all conjugate.

The group $R^*_{N=1}$ contains elements of finite order $4, 6$, and 14, from quadratic cyclotomic extensions of A.

The quotient $R^*_{N=1}/\langle \pm 1 \rangle$ contains elements of order $2, 3$, and 7, and is isomorphic to the triangle group $\Delta(2, 3, 7)$.

Let G be the algebraic group over k, with $G(k) = B^*/k^*$.

Let $M(1) = \prod_\lambda \mathrm{PGL}_2(A_\lambda) \subset G(\mathbb{A}^f)$ and define

$$X(1) = G(k)\backslash \mathfrak{H}^{\pm 1} \times G(\mathbb{A}^f)/M(1).$$

There is only one orbit for $G(k)$ on $G(\mathbb{A}^f)/M(1)$, with stabilizer conjugate to R^*/A^*.

$$X(1) = (R^*/A^*)\backslash \mathfrak{H}^{\pm 1} = (R^*_{N>0}/A^*)\backslash \mathfrak{H} = (R^*_{N=1}/\langle \pm 1 \rangle)\backslash \mathfrak{H}.$$

Arithmetic H-curves $X \to X(1) \cong \Delta(2,3,7)\backslash\mathfrak{H}$ come from normal subgroups of $M(1)$.

Let $\mathfrak{p}$ be a prime ideal of A, $F_{\mathfrak{p}} = A/\mathfrak{p}$, and $q = \#F_{\mathfrak{p}}$.

The finite group $\mathrm{PGL}_2(F_{\mathfrak{p}})$ is a quotient of $\mathrm{PGL}_2(A_{\mathfrak{p}})$ and acts triply transitively on the $q+1$ points of the projective line $\mathbb{P}^1(F_{\mathfrak{p}})$.

Consider the subgroups $M(\mathfrak{p}) \subset M_0(\mathfrak{p}^2) \subset M_0(\mathfrak{p}) \subset M(1)$ whose images in $\mathrm{PGL}_2(F_{\mathfrak{p}})$ fix three, two, and one point on $\mathbb{P}^1(F_{\mathfrak{p}})$.

Each gives a compact Riemann surface of the form

$$X(\mathfrak{p}) = G(k)\backslash\mathfrak{H}^{\pm 1} \times G(\mathbb{A}^f)/M(\mathfrak{p}).$$

We have maps

$$X(\mathfrak{p}) \longrightarrow X_0(\mathfrak{p}^2) \longrightarrow X_0(\mathfrak{p}) \longrightarrow X(1).$$

All four curves have **canonical models** over the cubic field $k = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, embedded in $\mathbb{C}$ by the real place v.

The curves $X(1), X_0(\mathfrak{p}), X_0(\mathfrak{p}^2)$ are geometrically connected.

The curve $X(\mathfrak{p})$ is is a Galois extension of $X(1)$ with group $\mathrm{PGL}_2(F_\mathfrak{p}) = \mathrm{PGL}_2(q)$. It is geometrically connected when $\mathfrak{p} = 2$ and has two components over $\mathbb{C}$ when q is odd.

The two components $X^{\pm}(\mathfrak{p})$ are rational over the unique quadratic extension K ot k of discriminant $\mathfrak{p}$. They give Galois coverings of $X(1)$ with group $\mathrm{PSL}_2(\mathfrak{p})$ and are H-curves.

The H-curves of genus $g = 3, 7, 14, 118$ correspond to prime ideals $\mathfrak{p} \subset A$ with $N\mathfrak{p} = 7, 8, 13, 27$.

When $\mathfrak{p} = 2A$, the Shimura curve $X(2)$ gives a canonical descent of the Fricke-Macbeath curve, together with its full automorphism group $G = \mathrm{PGL}_2(8)$, to $k = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$.

The quotient $X(2)/V$ by any Klein 4-group in $\mathrm{PGL}_2(8)$ is an elliptic curve E of conductor 2^2 over k.

This is the elliptic curve 64.1-a7 in LFMDB.

E descends uniquely to $\mathbb{Q}$, where it has minimal equation

$$y^2 = x^3 + x^2 - 114x - 127 \quad \Delta = 2^4.7^8 \quad j(E) = 2^8.7 = 1792.$$

Finally

$$J(X(2)) \sim E^7.$$

Automorphic forms

Recall the curves

$$X(\mathfrak{p}) \to X_0(\mathfrak{p}^2) \to X_0(\mathfrak{p}) \to X(1) = \Delta(2,3,7)\backslash\mathfrak{H}.$$

A holomorphic differential ω on $X_0(\mathfrak{p})$ or $X_0(\mathfrak{p}^2)$ which is an eigenvector for the Hecke correspondences corresponds to a new vector in an irreducible automorphic representation $\pi(\omega)$ of the group $G(\mathbb{A}) = (B \otimes \mathbb{A})^*/\mathbb{A}^*$.

Via a result of Jacquet-Langlands, $\pi(\omega)$ corresponds to a cuspidal automorphic representation $\pi = \otimes\pi_v$ of the split group $\mathrm{PGL}_2(\mathbb{A})$, which has conductor $\mathfrak{p}$ or $\mathfrak{p}^2$ and is in the discrete series of weight 2 at all three real places.

The action of $\mathrm{PGL}_2(F_\mathfrak{p})$ on the holomorphic differentials of $X(\mathfrak{p})$ helps to identify the local representation $\pi_\mathfrak{p}$ of $\mathrm{PGL}_2(k_\mathfrak{p})$.

Assume $\mathfrak{p} = 2A$. Then $X_0(2)$ has genus zero and $X_0(2^2)$ has genus one. The former is the projective line and the latter is the unique principal homogeneous space for $E = X(2)/V$ over k, which is non-trivial at the three real places and trivial elsewhere.

There is a unique cuspidal automorphic representation π of the split group $\mathrm{PGL}_2(\mathbb{A})$, with the following local behavior.

- At each real place v the representation π_v is the discrete series of weight 2 for $\mathrm{PGL}_2(\mathbb{R})$.
- At the prime $\mathfrak{p} = 2A$, the representation $\pi_\mathfrak{p}$ has conductor $\mathfrak{p}^2$. It is the discrete series of depth zero which is compactly induced from the representation $V_\mathfrak{p}$ of $\mathrm{PGL}_2(A_\mathfrak{p}) \to \mathrm{PGL}_2(F_\mathfrak{p})$ of dimension 7, associated to the two cubic characters $\{\chi_3, \chi_3^{-1}\}$ of the non-split torus.
- At all other places λ, the representation π_λ is unramified.

Let D be the **definite** quaternion algebra over k ramified at the three real places and the prime $\mathfrak{p} = 2A$, and let G^* be the algebraic group over k with points D^*/k^*.

There is a unique automorphic representation π^* of the adelic group $G^*(\mathbb{A})$ which is

- the trivial representation the compact group $G^*(\mathbb{R}) = \mathrm{SO}_3$ at each real place,
- the two dimensional representation $W_\mathfrak{p}$ of the compact group $G^*(k_\mathfrak{p})$ at the place $\mathfrak{p}$, which factors through its dihedral quotient of order 6 and whose restriction to the cyclic subgroup is the direct sum of the cubic characters $\{\chi_3, \chi_3^{-1}\}$,
- isomorphic to the unramified representation π_λ at all remaining finite places λ.

There is a unique invariant trilinear form on the local representations π_v^*, for **every** place v.

Hence there is a $G^*(\mathbb{A})$-invariant linear form on the automorphic representation $\pi^* \otimes \pi^* \otimes \pi^*$ of $G^*(\mathbb{A})^3$, which is unique up to scaling.

Define a linear form P on the automorphic representation $\pi^* \otimes \pi^* \otimes \pi^*$ of $G^*(\mathbb{A})^3$ by integration over the diagonal

$$P(\phi_1^* \otimes \phi_2^* \otimes \phi_3^*) = \int_{G^*(k)\backslash G^*(\mathbb{A})} \phi_1^*(g)\phi_2^*(g)\phi_3^*(g)\, dg.$$

The triple product L-function $L(\pi \otimes \pi \otimes \pi, s)$ satisfies a functional equation when $s \to 4 - s$, with sign $+1$.

The $G^*(\mathbb{A})$ invariant linear form P is **non-zero** if and only if

$$L(\pi \otimes \pi \otimes \pi, 2) \neq 0.$$

M. Harris and S. Kudla, ”The central critical value of a triple product L-function.” Annals of Math. **133** (1991).

We can test the non-triviality of P using a test vector $\phi^* = \prod \phi_v^*$ in $\pi^* \otimes \pi^* \otimes \pi^*$ for the invariant trilinear form.

The local vector ϕ_v^* spans the line fixed by an open compact subgroup M_v.

The test function ϕ^* is defined on the double coset space

$$G^*(k)^3 \backslash G^*(\mathbb{A})^3 / \prod M_v$$

which has **only five elements**.

One can identify ϕ^* using the Hecke operator T_λ with $N(\lambda) = 7$.

Integration over the diagonal is the evaluation of ϕ^* on one of the double cosets Δ, and $\phi^*(\Delta) \neq 0$.

Hence

$$L(\pi \otimes \pi \otimes \pi, 2) \neq 0.$$

Why is this computation so simple?

First, the quaternion algebra D is definite.

But more remarkable is the fact that, up to conjugacy, D contains a unique maximal A order, which has class number 1.

In fact, this order is

$$Ai + Aj + Ak + A(1 + i + j + k)/2 \qquad i^2 = j^2 = k^2 = ijk = -1.$$

cf. A. Hurwitz, "Vorlesungen Über die Zahlentheorie der Quaternionen" (1919).

The elliptic curve $E = J_0(2^2)$ is modular over k:

$$L(H^1(E), s) = L(\pi, s).$$

It follows that

$$L(\pi \otimes \pi \otimes \pi, s) = L(H^1(E)^{\otimes 3}, s) = L(\mathrm{Sym}^3 H^1(E), s)\, L(H^1(E), s-1)^2.$$

Since the triple product L-function does not vanish at $s = 2$

$$L(\mathrm{Sym}^3 H^1(E), 2) \neq 0 \quad L(H^1(E), 1) \neq 0.$$

The Jacobian $J(2)$ of the Fricke-Macbeath curve $X = X(2)$ is isogenous to E^7 over k, so we have an identity of motives of rank 14 over k: $H^1(X) = 7.H^1(E)$.

Let $Y = X^3$. The Kunneth formula gives an identity of motives of rank 2786 over k:

$$H^3(Y) = 343.\operatorname{Sym}^3 H^1(E) + 707.H^1(E)(-1).$$

Beilinson and Bloch suggest that the Chow group of codimension 2 cycles which are homologous to zero on a three-fold Y over k is finitely generated, of rank equal to the order of vanishing of the L-function of $H^3(Y)$ at the point $s = 2$.

In our case, when $Y = X^3$:

$$L(H^3(Y), s) = L(\operatorname{Sym}^3 H^1(E), s)^{343}\, L(H^1(E), s-1)^{707}$$

and the order of vanishing at $s = 2$ is zero.

Expectation Let $k = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ and let X be the Fricke-Macbeath curve over k. Then any codimension two cycle on $Y = X^3$ which is defined over k and is homologous to zero has finite order in the Chow group.

The modified diagonal cycle $\Delta X_e \to Y = X^3$ is defined over k and is homologous to zero.

Bisogno, Li, Litt, and Srinivasan have shown that the image of ΔX_e under the ℓ-adic Abel-Jacobi map is torsion.

Update: A few days after this talk, Wei Zhang and Congling Qiu sent me a proof that the modified diagonal cycle has finite order in the Chow group.

Their argument uses the geometric fact that there are no $\mathrm{SL}_2(8)$-invariant trilinear forms on the holomorphic differentials of X, and the arithmetic fact that the Mordell-Weil group of the Jacobian of X over k is finite.