Local Jiang's Conjecture for Arthur Packets

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(Joint with Baiying Liu)

Enhanced generic (tempered) L-Packet Conj.

G = quasisplit/F $\qquad F = p$-adic

Conj: An A-Packet is tempered iff it contains a generic member (Enhanced version of 1990 Annals Conj)

Harish-Chandra/Howe π = irr. adm. rep of $G(F)$

$\exists$ a n.b.h.d. U of $1 \in G(F)$ s.t. for $\forall f \in C_c^\infty(U)$

$$\mathrm{tr}(\pi(f)) = \sum_{\substack{\mathcal{O} = \text{nilpotent} \\ \text{orbit}}} c_{\mathcal{O}}(\pi)\, \hat{\mu}_{\mathcal{O}}(\tilde{f})$$

$$\hat{\mu}_{\mathcal{O}}(\tilde{f}) = \int_{\mathcal{O}} \hat{\tilde{f}}(u)\, d\mu_{\mathcal{O}}(u)$$

measure on the orbit

Fourier transform on Lie(G) $\qquad \tilde{f} = f \cdot \exp$

Wave front set of π is the set of nilpotent orbits $\mathcal{O}$ of highest dim. s.t. $c_{\mathcal{O}}(\pi) \neq 0$.

(2)

π = generic WFS = regular nilpotents (MW)

Thus the conj $\Longrightarrow$ Non-tempered A-packets

Cannot have regular WFS.

Jiang's conjecture suggests what they should be.

Arthur Packets

$G = Sp_{2n}$ $\quad {}^L G = SO_{2n+1}(\mathbb{C})$

$$\psi : W_F \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \longrightarrow SO_{2n+1}(\mathbb{C})$$

$$\psi = \bigoplus_{i=1}^{r} \phi_i \otimes S_{m_i} \otimes S_{n_i}$$

$\phi_i(W_F)$ = bdd, ss. $\dim \phi_i = k_i$.

$\sum_{i=1}^{r} k_i m_i n_i = 2n+1$ $\quad S_\ell = \ell$-dim. irr rep. of $SL_2(\mathbb{C})$

$$\psi \longmapsto \phi_\psi = \psi\left(w, x, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right)$$

non-tempered Langlands Parameter if $n_i > 1$

$$\phi_i(w) S_{m_i}(x) S_{n_i}\left(\begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right) = \sum_{j=-\frac{n_i-1}{2}}^{\frac{n_i-1}{2}} |w|^{j} \phi_i(w) S_{m_i}(x)$$

③

$\phi_\psi \longleftrightarrow \pi_\psi$ = irr. adm. unit self-dual rep of $GL_{2n+1}(F)$

$N \in \mathbb{N}$ $\quad \tilde{J} =$ 2nd diag$(1, -1, \ldots, (-1)^{N+1}) \in G(N)$

$G(N) = GL(N)$ $\quad \theta : G(N) \longrightarrow G(N)$

$\qquad\qquad g \longmapsto {}^t g^{-1}$

$$\tilde{\theta} = \mathrm{Int}(\tilde{J}) \cdot \theta : g \longmapsto \tilde{J}\,\theta(g)\,\tilde{J}^{-1}$$

Disconnected gp $\quad \tilde{G}^+(N) = G(N) \rtimes \langle \tilde{\theta} \rangle$

$$= G^0(N) \amalg \tilde{G}(N)$$

bitorsor $= G(N) \times \tilde{\theta}$

(π, V) = Irr. admissible self-dual rep. of $GL_N(F)$

$\exists!$ intertwining operator $\tilde{\pi}(N) : \pi \longrightarrow \tilde{\pi}$

Then π can be extended to a rep. of $\tilde{G}^+(N)$

and by restriction $\tilde{G}(N)$

$\psi \longleftrightarrow \phi_\psi \longleftrightarrow \pi_\psi$ = rep. $GL_{2n+1}(F)$

$\tilde{\pi}_\psi$ = ext. to bitorsor $\qquad N = 2n+1$

$\mathcal{H}(N)$ = Hecke alg of $G(N)$ = smooth, compactly supp.

$G = Sp_{2n}$ $\qquad$ $\mathcal{H}(G)$ = Hecke alg of $G(F)$

γ = strongly reg. s.s. conj. class in G
(abelian centralizer)

δ = " " " stable " "

$$\delta = \coprod \gamma \qquad \delta \to \gamma$$

$f \in \mathcal{H}(G)$ $\qquad$ f_G = normalized orbital integral of f
= $|\text{discriminant}|^{1/2}$ times integral over the orbit " conjugacy class of γ

$$f^G(\delta) := \sum_{\gamma \to \delta} f_G(\gamma)$$

$$S(G) = \{f^G \mid f \in \mathcal{H}(G)\}$$

Langlands - Shelstad - Kottwitz:

Given $\tilde{f} \in \mathcal{H}(N)$, there exists $f \in \mathcal{H}(G)$ such that $\tilde{f}$ and f have matching orbital integrals:

If the strongly regular class δ goes to a θ-conjugacy class $\tilde{\gamma}$ under the "image" map, δ is the norm of $\tilde{\gamma}$, then the normalized θ-twisted orbital integral $\mathrm{orb}_\theta(\tilde{f}, \tilde{\gamma}) = f^G(\delta)$.

(H = endoscopic for G) $\quad \mathcal{A}_{H/G} : Cl_{ss}(H)/F \longrightarrow Cl_{ss}(G,\theta)/F, \ \delta \longmapsto \tilde{\gamma}$

If δ is not a norm, we set $f^G(\delta) = 0$. Proved by Ngo and Waldspurger.

Thm (Arthur). For an $\tilde{f} \in \mathcal{H}(N)$, let

$$\tilde{f}_N(\psi) = \mathrm{tr}(\tilde{\pi}_\psi(\tilde{f}))$$

Then

$S(\psi)$ duality between S and $\tilde{\Pi}_\psi$

$$\tilde{f}_N(\psi) = \sum_{\pi \in \tilde{\Pi}_\psi} \langle s_\psi, \pi \rangle f_G(\pi),$$

$s_\psi \in S(\psi)$, $\tilde{\Pi}_\psi$ = Arthur Packet

$f_G(\pi) = \mathrm{tr}(\pi(f))$, $\tilde{f} \longleftrightarrow f$ and

$$s_\psi = \psi\left(1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$a_i = k_i m_i$, $n_i = b_i$, ϕ_i

$\phi_i \otimes S_{m_i} \rightsquigarrow \tau_i$ = tempered rep. of $GL_{a_i}(F)$

$$\bigoplus_i \phi_i \otimes S_{m_i} \otimes S_{n_i} \longleftrightarrow \bigoplus_i (\tau_i, b_i)$$

with (τ_i, b_i) Speh-type representations as quotients of:

$$\tau_i |\ |^{\frac{b_i - 1}{2}} \times \cdots \times \tau_i |\ |^{-\frac{b_i - 1}{2}}$$

May assume $b_1 \geq b_2 \geq \cdots \geq b_r$

$$\bigoplus (\tau_i, b_i) \longrightarrow [b_1^{a_1} \cdots b_r^{a_r}] = \text{partition of } 2n+1$$

⑥

$\psi \longleftrightarrow P(\psi) \in \mathcal{P}_1(2n+1) = \mathcal{P}(SO_{2n+1}(\mathbb{C}))$

Collingwood–McGovern

$P \in \mathcal{P}_1(2n+1)$, $\mathfrak{g} = \mathrm{Lie}(Sp_{2n})$ $\mathfrak{g}^\vee = \mathrm{Lie}(SO_{2n+1})$

Barbasch–Vogan duality (Achar)

$\eta = \eta_{\mathfrak{g}^\vee, \mathfrak{g}} : \mathcal{P}_1(2n+1) \longrightarrow \mathcal{P}(Sp_{2n})$

$\eta([b_1^{a_1} \cdots b_r^{a_r}]) := [([b_1^{a_1} \cdots b_r^{a_r}]^t)^-]$ collapse Sp

Young diag. $= \begin{bmatrix} a_1 \times b_1 \\ a_2 \times b_2 \\ \vdots \end{bmatrix}$ Young diag.

<u>Example</u> $P = [b^a] \in \mathcal{P}_1(2n+1)$

Then $ab = 2n+1$ and thus a and b are both odd. Now $\eta(P) = [((P^t)^-)_{Sp_{2n}}]$.

Note that $[b^a]^t = [a^b]$. Next

$$[a^b]^- = [a^{b-1}(a-1)].$$

Now $a-1$ and $b-1$ are both even and thus the odd parts, being a in this case, appear with even multiplicity, i.e., ⑦

$[a^{b-1}(a-1)]$ is already symplectic and thus no collapse is needed.

<u>Theorem</u>. <u>WFS of π_ψ is</u>

$$p^m(\pi_\psi) = \{p(\psi)^t\} \quad p(\psi) \in \mathcal{P}(2n+1)$$

We realize π_ψ as a full induced and get $\mathcal{P}^m(\pi_\psi)$ as an induced partition (orbit)

<u>Conj. 1 (Jiang).</u> <u>Given Arthur Parameter</u> ψ, <u>let</u>:

$$\widetilde{\Pi}_\psi = \underline{\text{local Arthur Packet}}$$

1) <u>For any</u> $\pi \in \widetilde{\Pi}_\psi$, <u>any</u> $p \in \mathcal{P}^m(\pi)$, <u>in the wave front set has the property</u>:

$$p \leq \eta_{\mathfrak{g}^\vee, \mathfrak{g}}(p(\psi)),$$

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where $\leq$ is the closure inclusion for orbits; equivalently the corresponding partition order.

2) There exists at least one $\pi \in \widetilde{\Pi}_\psi$ s.t.

$$\eta_{\mathfrak{g}^\vee, \mathfrak{g}}(p(\psi)) \in p^m(\pi)$$

Conj 2. $p^m(\widetilde{\pi}_\psi) = \{(b_1^{a_1} \cdots b_r^{a_r})^t_{SO_{2n+1}}\}$

Here $p^m(\widetilde{\pi}_\psi)$ is defined by character expansions for disconnected groups by Clozel / Konno, Varma.
(Compare with the WFS of π_ψ: Needs the Collapse)

Thm 1. Assume Conj 2 & $S_\psi = 1$. Then for any $p > \eta_{\mathfrak{g}^\vee, \mathfrak{g}}(p(\psi))$, $p \notin \bigcup_{\pi \in \widetilde{\Pi}_\psi} p^m(\pi)$

Thm 2. Assume Conj 2 + all b_i are odd ($\Longrightarrow S_\psi = 1$). Also assume nilpotent orbits in $p^m(\pi)$ belong to a unique orbit $/\bar{F}$
(Kawanaka / Moeglin-Waldspurger)

Then Conj 1 is valid.

Theorem 3. Assume $a_r = b_r = 1$ and b_i are all even for $1 \le i < r$. Assume Conj 2 is valid & orbits in $\mathfrak{p}^m(\pi)$ belong to a unique one $/\bar{F}$. If $S_\psi = 1$, then Jiang's Conjecture 1 is valid.

Outline of the proof

1. Construct a rep. σ of $Sp_{2n}(F)$ in the L-packet attached to ϕ_ψ, using some early work of Baiying Liu

1'. Determine WFP of π_ψ and $\tilde{\pi}_\psi$, rep. of $GL_{2n+1}(F)$ and $\widetilde{GL}_{2n+1}(F)$
 ↑ conjectural = conj. 2

2. Use the parity conditions to show that

$$\eta_{\mathfrak{g}^\vee, \mathfrak{g}}(P(\psi)) \in \mathfrak{p}(\sigma) = \text{Partitions (nilpotent orbits) of } \sigma \text{ for which } c_{\mathcal{O}}(\sigma) \neq 0$$

3. Need to show $\eta_{\mathfrak{g}^\vee, \mathfrak{g}}(P(\psi))$ is a wave front

Partition (Part 1 of Jiang's Conj.)

3a) Prove:

$$\dim_{Sp_{2n}}(\eta_{SO_{2n+1},Sp_{2n}}(P(\psi))=\dim_{SO_{2n+1}}(P(\psi)^t_{SO_{2n+1}})$$

3b) Use Arthur char. id. and arguments from 1990 - Annals paper, 3a, and Conj. 2 for bitorsor, to show $\eta_{\mathfrak{g}^\vee,\mathfrak{g}}(P(\psi))$ is in the wave front set, modulo the possibility of non-related maximal orbits.

But non-related orbits should not happen by uniqueness of WFS / F.

Example: $b_1=3$, $a_1=1$, $\tau_1=\mathbb{1}_{GL_1}$, $b_j=1$, $2\le j\le r$, and Conj. 2 is valid. Then Jiang's conj. is valid.

Pf. $P(\psi)=[3\,1^{2n-2}]$ $\eta(P(\psi))=[(P(\psi)^t)^-]_{Sp_{2n}}$

$[3\,1^{2n-2}]^t = [(2n-1)1^2]$ $\quad [(2n-1)1^2]^- = [(2n-1)1]$

$[(2n-1)1]_{Sp_{2n}} = [(2n-2)\,2]$ = sub-regular in Sp_{2n}

we only need to show no generic member appears in our non-tempered A-packet. But that follows from the enhanced generic packet conjecture which follows from conj 2.

Example. when parity is different on the partition: $[9^3 8^2 7^3 5^3 2^2]^t = [13^2\, 11^3\, 8^2\, 5\, 3]$

$[13^2\, 11^3\, 8^2\, 5\, 3]^- = [13^2\, 11^3\, 8^2\, 5\, 2]$

$[13^2\, 11^3\, 8^2\, 5\, 2]_{Sp} = [13^2\, 11^2\, 10\, 8^2\, 6\, 2] = \eta(\cdots)$

The one coming from our calculation via σ gives: $[13^2\, 11^2\, 8^3\, 5^2]^{Sp} :=$ Special orbit dominating

$[13^2\, 11^2\, 8^3\, 5^2] = [13^2\, 11^2\, 8^3\, 6\, 4] < \eta([9^3 8^2 7^3 5^3 2^2])$

Main references beside those mentioned earlier: Gomez-Gourevitch-Sahi (2017); Jiang-Liu-Savin (2016)

D. Jiang / Cotemprorary Math. 614 - Piatetski volume

HAPPY BIRTHDAY STEVE!