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Representations of reductive p-adic groups over number or finite fields

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HAPPY BIRTHDAY STEVE

I met Steve, at the summer school in Bonn in 1976 (Modular Functions of One Variable) a year before Corvallis (Automorphic Forms, Representations, and L-functions).

In the lecture notes of the conference we both published a paper.
K. *Relations between automorphic forms produced by theta-functions*
V. *Séries thêta des formes quadratiques indéfinies*

40 years later, we were both invited to the 2017 Banff worshop on Automorphic Forms, Mock Modular Forms and String Theory.

Steve spoke on his beautiful work with Funke related to their former work with Millson
Mock modular forms and geometric theta functions for indefinite quadratic forms.

I was invited because of my Bonn paper used by Alexandrov, Banerjee, Manschot, Pioline *Indefinite theta series and generalised error functions*.

But I had totally forgotten it and I had nothing to say related to the thema of the conference ... By miracle two weeks before the conference, I read the computations of Gan and Sawin on minimal representations (they are related to theta series), and I saw how to use them to show the existence of *supersingular mod p representations of many reductive p-adic groups* which was an open question. I could give a talk on *minimal representations of p-adic groups* with a new result.

This tiny thread relates indefinite theta series my talk to-day !

Part I: Basic facts, Part II: Cuspidal types (with Henniart),
Part III: Asymptotics of dimensions of invariants (in progress)

I

F a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, so $\mathrm{char} F = 0$ or p
G group of rational points of a connected reductive F-group
C a field of characteristic c and prime subfield $\mathbb{F}_c$, so $\mathbb{Q} = \mathbb{F}_0$!

We recall that a C-representation π of G is called
smooth when every vector is fixed by an open subgroup of G.
admissible when it is smooth and for all compact open subgroups K of G, the C-vector space π^K of K-invariant vectors has finite dimension.
$\mathrm{Irr}_C(G)$ family of irreducible admissible C-representations of G.

subquotient of an admissible is admissible if $(\mathrm{char} F$ or $\mathrm{char} C) \neq p$
Counter-example when $\mathrm{char} C = \mathrm{char} F = p$ (Abe-Henniart-V)
Irreducible implies admissible if $\mathrm{char} C \neq p$
Counter-example when $\mathrm{char} C = p$ (D. Le)

The category $\mathrm{Mod}_C(G)$ of smooth C-representations of G is abelian and Grothendieck.

P parabolic subgroup of G, unipotent radical N, Levi subgroup $M \simeq P/N$

Parabolic induction $\mathrm{Ind}_P^G : \mathrm{Mod}_C(M) \to \mathrm{Mod}_C(P) \to \mathrm{Mod}_C(G)$ inflation followed by smooth induction,

$\sigma \to \tilde{\sigma} \to \{f : G \to \tilde{\sigma},\ f(pgk) = \tilde{\sigma}(p) f(g)\}$ k in some open subgroup, G acts by right translation.

Left adjoint $L_P^G = (-)_N$, right adjoint R_P^G

Second adjunction conjecture $R_P^G = L_{\overline{P}}^G \rho_P$ when $\mathrm{char}\, C \neq p$

ρ_P modulus of P, $\overline{P}$ opposite parabolic subgroup of P.

Proved by Bernstein when $C = \mathbb{C}$, by Dat when $G = GL(n)$ or a classical group if $p \neq 2$, or a moderately ramified group if $(p, W_{\underline{G}}) = 1$.

Finite group $R_P^G = (-)^N$, $L_P^G \simeq R_P^G$.

$\pi \in \mathrm{Mod}_C(G)$ non zero is called
left cuspidal if $L_P^G \pi = 0$ for all $P \neq G$.
right cuspidal if $R_P^G \pi = 0$ for all $P \neq G$,
cuspidal if left and right cuspidal.

These distinctions are necessary when $\mathrm{char}\, C = p$, for example the trivial representation 1 not left cuspidal but right cuspidal.
When $\mathrm{char}\, C \neq p$, the second adjunction conjecture implies that left cuspidal=right cuspidal. But for π irreducible one can prove - without using the conjecture that left cuspidal = right cuspidal

$\pi \in \mathrm{Irr}_C(G)$ is called
supercuspidal if not isomorphic to a subquotient of $\mathrm{ind}_P^G \sigma$, $P \neq G$, $\sigma \in \mathrm{Irr}_C(M)$
supercuspidal $\Rightarrow$ cuspidal.

cuspidal = supercuspidal when $\mathrm{char}\, C = 0, p, \ell$ big with respect to G (banal).
Counter-example: G=$GL(2, F)$ and $\mathrm{char}\, C = \ell$ divides $q + 1$.

Steinberg representation $\mathrm{St} = \mathrm{ind}_B^G 1 / \sum_P \mathrm{ind}_P^G 1$
St irreducible when $\mathrm{char}\, C = 0, p, \ell$ banal
Counter-example: G=$GL(2, F)$ and $\mathrm{char}\, C = \ell$ divides $q + 1$,
St has length 2, $\pi \subsetneq \mathrm{St}$ cuspidal not supercuspidal

$\pi \in \mathrm{Irr}_C(G)$ restricted to the center Z of G is ω-isotypic for some $\omega \in \mathrm{Irr}_C(Z)$, and it is often convenient to work in the Grothendiieck abelian category $\mathrm{Mod}_C(G, \omega)$ of smooth C-representations of G with ω-isotypic restriction to Z.
$\mathrm{char}\, C \neq p$, π supercuspidal = injective envelope $I_\pi \in \mathrm{Mod}_C(G, \omega)$ is right cuspidal (Hiss for finite reductive groups)

II

When $\pi \in \mathrm{Irr}_{\mathbb{C}}(G)$ is cuspidal, it is conjectured that π is compactly induced from an open subgroup of G compact modulo the center. We proved with Henniart that when this conjecture has been proved, then it is true for any field C of characteristic $\neq p$!

Compact induction: J is an open sg of G containing Z the centre and compact modulo Z, $\lambda \in \mathrm{Mod}_C(J)$,
$\mathrm{ind}_J^G(\lambda) = \{f : G \to \lambda,\ f(jg) = \lambda(j)f(g), \text{support compact mod} J\}$
A cuspidal C-type in G is a pair (J, λ) such that $\mathrm{ind}_J^G \lambda$ is irreducible.

If $\mathrm{ind}_J^G \lambda$ is irreducible then $\mathrm{char} C \neq p$ and $\mathrm{ind}_J^G \lambda$ is cuspidal.

A cuspidal C-type (J, λ) in G satisfies intertwinning if $\mathrm{End}_{C[J]}(\lambda) \simeq \mathrm{End}_{C[G]}(\mathrm{ind}_J^G \lambda)$ (automatic if $C = \mathbb{C}$ or $C = C^a$).

$\mathrm{Aut}(C)$ acts on the family of cuspidal C-types (J, λ) in G satisfying intertwinning, When $\mathrm{char}\, C \neq p$, there exists an explicit list $\mathfrak{X}$ of cuspidal C-types (J, λ) in G satisfying intertwinning, stable under $\mathrm{Aut}(C)$, such that any irreducible cuspidal C-representation of G of level 0 is compactly induced from some $(J, \lambda) \in \mathfrak{X}$ (exhaustion) unique modulo G-conjugation (unicity).

The theorem is true withoout level 0 for the following groups G:
The semisimple rank of G is ≤ 1 (except for unicity, not known for all G)
$G = SL(n, F)$
$G = GL(n, D)$ for a central division algebra D of finite F-dimension
G is a classical group as in Shaun Stevens or a quaternionic form of such a group, and $p \neq 2$
G is moderately ramified and p does not divide the order of the absolute Weyl group of G.

When C is algebraically closed, an explicit list satisfying EU is given by Weissman (rank 1), Cui ($SL(n, F)$), Bushnell-Kutzko, Minguez-Secherre ($GL(n, D)$), Kurinczuk-Skodlerack-Stevens (G classical), Skodlerak (quaternionic form), Fintzen-Yu (moderately ramified) (Unicity: Hakim-Murnaghan, Deseine).
We check that these lists are $\mathrm{Aut}(C)$-stable.

Let C'^a be an algebraically closed field and C a field of the same characteristic. Any list of cuspidal C'^a-types satisfying EUS in G gives a list of cuspidal C-types in G satisfying EUSI in three steps:

1) Go from C'^a to C^a via $\mathbb{F}_c^a$, ($\mathbb{F}_0 = \mathbb{Q}$ if $c = 0$)

Base change yields bijections
irr.cusp. $\mathbb{F}_c^a$-repr. $\leftrightarrow$ irr.cusp. C^a-repr. with finite order central character
Lists of cuspidal $\mathbb{F}_c^a$-types in G satisfying EUS $\leftrightarrow$ lists of cuspidal C^a-types in G satisfying EUS with finite order central character

2) Take G-normalizers

Replace each (J,λ) in a list of cuspidal C^a-types in G satisfying EUS by its G-normaliser $(K = N_G(J), \rho = \mathrm{ind}_J^K \lambda)$. Get a list of cuspidal C^a-types (K,ρ) in G satisfying EUS and $\mathrm{Aut}\, C^a$-unicity: $\sigma \in \mathrm{Aut}\, C^a$, $\sigma(\mathrm{ind}_K^G \rho) \simeq \mathrm{ind}_K^G \rho$ implies $\sigma(\rho) \simeq \rho$.

3) Descent from C^a to C.

Replace each (J,λ^a) in a list cuspidal C^a-types in G satisfying EUS and $\mathrm{Aut}\, C^a$-unicity by (J,λ) for $\lambda \in \mathrm{Irr}_C(J)$ such that as a C-representation, λ^a is a direct sum of copies of λ. Get a list of cuspidal C-types in G satisfying EUSI.
This follows from a theorem of general algebra describing $C^a \otimes_C \pi$ for a simple module π over a C-algebra with $\dim_C \mathrm{End}\pi < \infty$.

There is a notion of supercuspidal C-type for the cuspidal C-types (J, λ) in G of level 0 or constructed as above. A cuspidal C^a-representation of a finite reductive group appears in the construction, (J, λ) is supercuspidal if the irreducible components of this representation are supercuspidal.

If (J, λ) is supercuspidal then $\mathrm{ind}_J^G \lambda$ is supercuspidal.
The converse is true for (J, λ) of level 0, and for any (J, λ) if (G, C) satisfies the second adjunction ($G = GL(n, F)$, a classical group if $p \neq 2$, a moderately ramified group if p does not divide the order of the Weyl group).

The proof uses the characterisation of supercuspidality of an irreducible C-representation by the right cuspidality of its injective enveloppe.
The theorem was not known when C is algebraically closed,

Reference: Guy Henniart, V. ArXiv Nov 2020 2010.0642

III

$\mathrm{char}\, C = p$

We take $\pi \in \mathrm{Irr}_C(G)$. Then $\pi^{I_1} \neq 0$ (pro-p Iwahori subgroup I_1).

When a certain central element of the pro-p-Iwahori ring $\mathbb{Z}[I_1 \backslash G / I_1]$ kills π^{I_1}, π is called supersingular (terminology introduced by Barthel-Livne).

cuspidal= supercuspidal = supersingular
(C alg.closed Abe-Henniart-Herzig-V, Ollivier-V.)

Irreducible admissible supersingular $\mathbb{F}_p^a$-representations $GL(2, \mathbb{Q}_p)$ are classified (Barthel-Livne-Breuil)

Existence for all G if $\mathrm{char} F = 0$ (the link with theta series) Herzig-Koziol-V.

S. Morra: *Invariant elements for p-modular representations of $GL_2(Q_p)$*
$G = GL(2, \mathbb{Q}_p)$, $\pi \in \mathrm{Irr}_{\mathbb{F}_p^a}(G)$ of infinite dimension

$$\dim_{\mathbb{F}_p^a} \pi^{1+p^n M(2,\mathbb{Z}_p)} = c_0 + c_1 p^n, \quad n \geq 1$$

$c_1 = 1 + p^{-1}, c_0 = 0$ if π is a principal series,
$c_1 = 1 + p^{-1}, c_0 = 1$ if π is a Steinberg representation,
$c_1 = 2(1 + p^{-1}), c_0 = -3$ or -4 if π is supersingular

Breuil Herzig Hu Morra Schraen: *Gelfand-Kirillov dimension and mod p cohomology for GL_2*
$G = GL(2, F)$, $F/\mathbb{Q}_p$ unramified degree f, most admissible $\pi \in \mathrm{Mod}_{\mathbb{F}_p^a}(G)$ coming from mod p-cohomology satisfy: there exists $a, b \in R_{>0}$ such that

$$a \leq p^{-fn} \dim_{\mathbb{F}_p^a} \pi^{1+p^n M(2,O_F)} \leq b, \quad n \geq 1$$

Compare with the theorem when $\mathrm{char}\, C \neq p$ (Howe when $C = \mathbb{C}$, proved when π is cuspidal):

Assume $\mathrm{char}\, C \neq p$, $G = GL(m, F)$, $m \geq 2$, $\pi \in \mathrm{Irr}_C(G)$. There exists a polynomial $P_\pi(X) = c_0 + c_1 X^{d_1} + \ldots + c_r X^{d_r} \in \mathbb{Z}[X]$, with a positive dominant coefficient, $0 = d_0 < d_1 < \ldots < d_r$ are the dimensions of the nilpotent G-orbits in $M(n, F)$ such that for large n,

$$\dim_C \pi^{1+p_F^{2n} M(m, O_F)} = P_\pi(q^n), \quad n >> 0$$

(q is the order of the residue field of F, p_F is a uniformizer of F)

For $G = GL(2, F)$, $\pi \in \mathrm{Irr}_C(G)$ of infinite dimension,

$$\dim_C \pi^{1+p^n M(2, O_F)} = c_0 + c_1 q^n, \quad n >> 0$$

$c_0 \in \mathbb{Z}$, $c_1 \in \mathbb{N}_{>0}$.

It is time to stop, as this is work in progress.

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