Ramification of supercuspidal parameters

work in progress with Gan and Sawin

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Outline

No theta series

This talk is about representation theory, geometry, and arithmetic but not about theta series.

The moral will be that a very simple application of a deep theorem in geometry has surprising consequences for representation theory specifically, for the representation theory of groups over local fields of positive characteristic p.

Even more specifically, the applications are consequences of the expected local Langlands correspondence. (What the expected LLC is will be revealed in the next slides.)

Nevertheless, my obsession with the LLC was an outgrowth of my work with Steve on the theta correspondence, though not directly. [Explain] [That's a stage direction]

What is the local Langlands conjecture?

The surprise answer is that there is no local Langlands conjecture.

Or else, (what comes to the same thing) there are several local Langlands conjectures.

In what follows G is a connected reductive group over a local non-archimedean field F of with residue field $k = \mathbb{F}_q$ of characteristic p; $\Pi(G/F)$ denotes the set of irreducible admissible representations of $G(F)$ with coefficients in C, an algebraically closed field of characteristic zero, with a chosen $q^{\frac{1}{2}}$. A Langlands parameter (L-parameter) for G/F is a pair

$$\rho : W_F \to {}^L G(C), N \in Lie(\hat{G})$$

with W_F the Weil group of F, satisfying the usual relations. The set of equivalence classes of L-parameters for G/F is denoted $\Phi(G/F)$.

First version of LLC

Conjecture

(a) *There is a canonical parametrization*

$$\mathcal{L} = \mathcal{L}_{G/F} : \Pi(G/F) \to \Phi(G/F).$$

(b) *For any $\varphi \in \Phi(G/F)$, the L-***packet** *$\Pi_\varphi := \mathcal{L}^\varphi$ is finite.*

(c) *For any $\varphi \in \Phi(G/F)$ the L-packet Π_φ is non-empty.*

The problem is: whose canon? As André Weil might have said, both sides have the same cardinality.
As stated, this is a vacuous conjecture.
In what follows we only state the conjecture for the subset $\Pi^{temp}(G/F) \subset \Pi(G/F)$, and will sometimes forget the superscript.

Some conditions

Property

When G is a torus, the map $\mathcal{L}$ is given by class field theory.

Property (Change of field)

Let $\alpha : F \to F'$ be an isomorphism of fields. Denote by α_* the identification of $\Phi(G/F)$ and $\Phi(G/F')$ corresponding to the canonical isomorphism $WD_F \to WD_{F'}$; likewise for $\Pi(G/F) \xrightarrow{\sim} \Pi(G/F')$. Then the following diagram is commutative.

$$\begin{array}{ccc} \Pi(G/F) & \xrightarrow{\mathcal{L}_{G/F}} & \Phi(G/F) \\ \alpha_* \downarrow & & \downarrow \alpha_* \\ \Pi(G/F') & \xrightarrow[\mathcal{L}_{G/F'}]{} & \Phi(G/F') \end{array}$$

More conditions

(twist) Let $\nu : G \to T$ be a homomorphism of algebraic groups over F, T a torus; $\nu^* : {}^L T \to {}^L G$ the dual map. Given $\pi \in \Pi(G/F)$, $\chi \in \Pi(T/F)$:

$$\mathcal{L}(\pi \otimes \chi \circ \nu) \xrightarrow{\sim} \mathcal{L}(\pi) \cdot \mathcal{L}(\chi) \circ \nu^*.$$

(contragredients) $\sigma \circ \mathcal{L}(\pi^\vee) \xrightarrow{\sim} (\sigma \circ \mathcal{L}(\pi))^\vee$
for any homomorphism $\sigma : {}^L G \to GL(N)$ of algebraic groups.

(Central isogenies) Let $r : G \to G'$ be a central isogeny, $r^* : {}^L G' \to {}^L G$ the dual map. Then if $\pi \in \Pi(G'/F)$,

$$\mathcal{L}(\pi \circ r) = r^* \circ \mathcal{L}(\pi).$$

Still more conditions

- Compatibility with products $G = H_1 \times H_2$.
- Compatibility with normalized parabolic induction
- Compatibility with restriction of scalars
- Compatibility with the Satake correspondence for spherical representations
- Suppose $\pi \in \Pi(G/F)$ belongs to the discrete series – for example, if π is supercuspidal. Then the quotient $Z_{^LG(C)}(\mathcal{L}(\phi))/Z(\hat{G}(C))^\Gamma$ is finite.

This long list still does not get anywhere near a characterization of $\mathcal{L}$.

Automorphic conditions

(i) Suppose $\sigma : {}^LG \to GL(N)$ is an algebraic representation. Suppose there is a theory of automorphic L-functions for G over F and σ (e.g. Rankin-Selberg or Langlands-Shahidi). Then for any $\pi \in \Pi(G/F)$ and $\psi : F \to \mathbb{C}^\times$,

(ii) Compatibility with endoscopic transfer.

(ii') Compatibility with cyclic base change.

(iii) Local global compatibility

Point (i) (for Rankin-Selberg for $GL(n) \times GL(m)$), together with some of the earlier conditions, suffice to characterize LLC for $GL(n)$ (Henniart). Point (i) is useless for E_8!

More on local-global compatibility

Point (iii) makes sense for all groups in positive characteristic, where the global parametrization is given by V. Lafforgue.
It has been proved by Genestier and Lafforgue, **up to semisimplification**.
But the original conditions are unknown for general G:

(a) There is a canonical parametrization

$$\mathcal{L} = \mathcal{L}_{G/F} : \Pi(G/F) \to \Phi(G/F).$$

(b) For any $\varphi \in \Phi(G/F)$, the ***L*-packet** $\Pi_\varphi := \mathcal{L}^\varphi$ is finite.

(c) For any $\varphi \in \Phi(G/F)$ the L-packet Π_φ is non-empty.

The results with Gan and Sawin do not (yet) help with (b) and (c), but they suggest a few directions.
More on this below.

Fargues-Scholze

For p-adic fields, the Fargues-Scholze construction replaces the Genestier-Lafforgue parametrization.

However, there is (in general) no corresponding global correspondence for automorphic representations, so no local-global compatibility.

Points (b) (finiteness of packets) and (c) (surjectivity of the parametrization) are not known for either Fargues-Scholze or Genestier-Lafforgue. In principle it's possible that infinitely many supercuspidals have **trivial** parameters in either case. This, at least, we can exclude by our methods (in most cases).

Kaletha's parametrization

Kaletha has used the construction of J.-K. Yu to define a partial parametrization $\mathcal{L}$ whose domain is the set of regular and non-singular supercuspidals representations of any $G(F)$. By Fintzen's results, this includes all supercuspidals if p is not too small for G.
Together with Fintzen and Spice, Kaletha recently proved character formulas that imply compatibility with endoscopic transfer and characterize the parametrization uniquely (except for small p).

Question

Is Kaletha's parametrization the same as Fargues-Scholze or Genestier-Lafforgue?

I don't know whether or not this is accessible by (Lefschetz) trace formula techniques.

The Deligne-Kazhdan correspondence

Let $F = k((t))$ be a local field of characteristic p and $F^\sharp$ a p-adic field that is *n-close* to F for some $n >> 0$:

$$\mathcal{O}_F/m_F^n \xrightarrow{\sim} \mathcal{O}_{F^\sharp}/m_{F^\sharp}^n$$

Then (Deligne) letting Φ^n denote Weil group parameters of depth n,

$$\Phi^n(G/F) \xrightarrow{\sim} \Phi^n(G/F^\sharp)$$

Let G be split connected, $K_r(F) \subset G(\mathcal{O}_F)$ (resp. $K_r(F^\sharp) \subset G(\mathcal{O}_{F^\sharp})$) the standard m_F^r-congruence subgroup.
Let $H(G(?), r) = H(G(?), K_r(?))$, $? = F, F^\sharp$. Then (Kazhdan) if $F^\sharp$ and F are n close for some $n >> r$ then letting Π^r denote irreducible representations of depth r.

$$\Pi^r(G/F) \xrightarrow{\sim} \Pi^r(G/F^\sharp).$$

An exercise

Conjecture

If $n >> r$ the following diagram commutes:

$$\begin{array}{ccc}
\Pi^r(G/F) & \xrightarrow{\textit{Genestier-Lafforgue}} & \Phi^n(G/F) \\
\Big\downarrow \textit{Kazhdan} & & \Big\downarrow \textit{Deligne} \\
\Pi^r(G/F^\sharp) & \xrightarrow[\textit{Fargues-Scholze}]{} & \Phi^n(G/F^\sharp)
\end{array}$$

Challenge

Prove this conjecture.

Review of V. Lafforgue's global results

Let Y be a smooth projective curve over $k = \mathbb{F}_q$, $\ell \neq p$ a prime.
G split semisimple over $K = k(Y)$.
$\mathcal{A}_0(G) = \{$ cuspidal automorphic representations of $G(\mathbf{A}_K)\}$,
$\mathcal{G}^{ss}(G) = \{$ semisimple maps $\rho_\ell : Gal(K^{sep}/K) \to \hat{G}(\overline{\mathbb{Q}}_\ell).\}$

Theorem (VL)

There is a map

$$\mathcal{L} : \mathcal{A}_0(G) \to \mathcal{G}^{ss}(G)$$

with the following property: if v is a place of K and $\Pi \in \mathcal{A}_0(G)$ is a cuspidal automorphic representation such that Π_v is unramified, then $\mathcal{L}(\Pi)$ is unramified at v, and the semisimplification $\mathcal{L}^{ss}(\Pi)\,|_{W_{K_v}}$ is the Satake parameter of Π_v.

Local parameters

Theorem (Genestier-Lafforgue)

With the above hypotheses, let w be any place of K. Then

$$\mathcal{L}_w(\Pi_w) := [\mathcal{L}(\Pi) \mid_{W_{K_w}}]^{ss}$$

depends only on K_w and Π_w (not on the globalizations K and Π). Moreover, $\mathcal{L}_w$ is compatible with parabolic induction in the obvious sense.

In particular, if $F = k((t))$ is an equal characteristic local field and π is an irreducible representation of $G(F)$, we can define the semisimple homomorphism

$$\mathcal{L}(\pi) : W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell).$$

Weights

Let $\sigma : \hat{G} \to GL(N)$ be any representation, S the set of primes where Π is ramified. Then

$$\mathcal{L}(\Pi)_\sigma := \sigma \circ \mathcal{L}(\Pi) : Gal(K^{sep}/K) \to GL(N, \overline{\mathbb{Q}}_\ell)$$

corresponds to a semi-simple ℓ-adic local system $L(\Pi)_\sigma$ on $Y \setminus |S|$.

By Deligne's Weil II, each irreducible summand of $L(\Pi)_\sigma$ is punctually pure (up to twist by a character of $Gal(\bar{k}/k)$).
Hence for any w, the eigenvalues of $\sigma \circ \mathcal{L}_w(\Pi_w)(Frob_w)$ are Weil q-numbers of various weights (up to the twist, which we ignore).

Say a representation π of $G(F)$ is *pure* if for some (equivalently, for any) faithful σ, all the eigenvalues of $\sigma \circ \mathcal{L}(\pi)(Frob_q)$ have the same weight.

What about supercuspidals?

If $G \neq GL(n)$, not all supercuspidals are pure. Here is our main theorem. Recall that G is split semisimple.

Theorem (GHS)

Let π be a **pure** *supercuspidal representation of $G(F)$. Suppose π is compactly induced from a compact open subgroup of G. (For example, if p does not divide the order of the Weyl group $W(G)$, this follows from* **Fintzen's theorem***.) Suppose moreover that $q > 3$. Then $\mathcal{L}(\pi)$ is not unramified.*

Henceforward we assume $p \nmid |W(G)|$. Because $\mathcal{L}$ is compatible with parabolic induction, we conclude

Corollary

Let π be a pure representation of $G(F)$. Suppose $\mathcal{L}(\pi)$ is unramified. Then π is an irreducible constituent of an unramified principal series.

Incorrigible representations

Assume local and global *cyclic stable base change* over function fields works as in Labesse's book. An *incorrigible representation* of $G(F)$ is a supercuspidal representation π such that, for any sequence $F \subset F_1 \subset \cdots \subset F_r$ of cyclic Galois extensions, the base change of π to F_r (which is an L-packet) contains a supercuspidal member.

Corollary. No pure supercuspidal representation is incorrigible.

Two proofs for $G = GL(n)$: Henniart (numerical correspondence), and Scholze (nearby cycles). Along with the existence of a canonical parametrization, this is **the** key step in any proof of LLC for $GL(n)$.
Last month Siyan Daniel Li Huerta (a third year student at Harvard) gave a new proof for $GL(n)$ in positive characteristic, modeled in Scholze's proof for p-adic fields.
The above corollary is again the key step.

Wild ramification

Theorem

Suppose π is a pure supercuspidal compactly induced from an open compact subgroup that is **small** *in an appropriate sense. Then $\mathcal{L}(\pi)$ is wildly ramified.*

An example of small open compact subgroup is the principal congruence subgroup $G(\mathcal{O}_F)_+ \subset G(\mathcal{O}_F)$.

Mixed supercuspidals

The supercuspidals π that are not pure have Langlands parameters (ρ, N) with $N \neq 0$. Then we expect that the pair (ρ, N) satisfies **purity of the monodromy weight filtration**: (up to unramified twist)

(i) For any if $\sigma : \hat{G} \rightarrow GL(M)$, the eigenvalues of $\sigma \circ \rho(Frob_F)$ are all q-numbers of integer weight.

(ii) The subspace $W_a V \subset V$ of eigenvectors for $\sigma \circ \rho(Frob_F)$ with eigenvalues of weight $\leq a$ is invariant under (ρ, N);

(iii) Letting $gr_a V = W_a V / W_{a-1} V$, there is $w \in \mathbb{Z}$ such that, for all $i \geq 0$, the map

$$N : gr_{w-i} V \rightarrow gr_{w+i} V$$

is an isomorphism.

Mixed supercuspidals

The parameter $\mathcal{L}(\pi)$ does not see N. However, not every semisimple parameter can be completed to one satisfying purity of MWF.
For example, it $\sigma \circ \rho$ has two weights a and a', and if it ρ can be completed to (ρ, N) satisfying purity, then $a \equiv a' \pmod 2$. We prove:

Theorem

For any supercuspidal π, $\mathcal{L}(\pi)$ can be completed to a pair $(\mathcal{L}^{ss}(\pi), N)$ that satisfies purity of the monodromy weight filtration.

We are thus entitled to define this pair to be the full Langlands parameter of π.

A theorem of Dat-Lanard

Theorem (Dat-Lanard, in progress)

Let π be a depth zero supercuspidal representation. (Maybe suppose $p \neq 2$.) Then $\mathcal{L}(\pi)$ is **tamely ramified**.

A depth zero representation is (roughly) compactly induced from a Deligne-Lusztig representation of the Levi quotient of a maximal parahoric subgroup. The property of wild ramification is preserved under reduction modulo any $\ell \neq p$. The idea of the proof is to use congruences of Deligne-Lusztig representations and induction on $\dim G$.

Globalization

Let $Y = \mathbb{P}^1$, $K = k(t)$. Choose an Borel $B \subset G$ (over k), B_- an opposite Borel. Let $I_0 \subset G(K_0)$ (resp. $I_{\infty,+} \subset G(K_\infty)$ denote the Iwahori corresponding to B (resp. the pro-unipotent radical of the Iwahori corresponding to B_-). We construct a cuspidal automorphic representation Π of $G(\mathbf{A}_K)$ such that

(a) At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k) \subset \mathbb{P}^1(\bar{k})$, $\Pi_z \xrightarrow{\sim} \pi$;

(b) For $x \notin |\mathbb{P}^1(k)|$, Π_x is unramified

(c) $\Pi_\infty^{I_{\infty,+}} \neq 0$.

(d) $\Pi_0^{I_{0,+}}$ contains a vector transforming under a certain character χ_k of $I_0/I_{0,+}$.

Here $I_{0,+} \subset I_0$ is what you think it is.

Application of purity

Suppose $\mathcal{L}(\pi)$ is unramified. By the purity hypothesis, for any faithful $\sigma \in Rep(\hat{G})$, $\mathcal{L}(\Pi)_\sigma$, which is a priori an ℓ-adic local system on $\mathbb{P}^1 \setminus |\mathbb{P}^1(k)|$, extends to a punctually pure local system on $\mathbb{G}_m$. (There is no unipotent monodromy at the points in $\mathbb{G}_m(k)$.) Moreover, our hypotheses imply that the ramification at 0 and ∞ is *tame*. Thus it is a sum of local systems induced from characters of finite order of the tame fundamental group of $\mathbb{G}_m$. Of course, $\mathcal{L}(\Pi)_z = \mathcal{L}(\pi)$ for every $z \in \mathbb{G}_m(k)$. By varying the character χ_k, we obtain a contradiction.

Poincaré series

Suppose for simplicity π is compactly induced from $U = G(\mathcal{O}_F)$. Let φ_π be a matrix coefficient of π supported in U, $\varphi_\pi(1) = 1$.
We construct Poincaré series on $G(\mathbf{A}_K)$ as in the Gan-Lomelí paper

$$P_\varphi(g) = \sum_{\gamma \in G(K)} \varphi(\gamma \cdot g), g \in G(\mathbf{A}_K)$$

where $\varphi = \prod_x \varphi_x$ with

(a) At every $z \in \mathbb{G}_m(k) \subset \mathbb{P}^1(k)$, $\varphi_z = \varphi_\pi$;
(b) For $x \notin |\mathbb{P}^1(k)|$, $\varphi_x = 1_{G(\mathcal{O}_x)}$;
(c) $\varphi_\infty = 1_{I_{\infty,+}}$;
(d) $\varphi_0 = \chi_k : I_0/I_{0,+} \to \mathbb{C}^\times$

The support conditions imply $P_\varphi(1) = 1$; then set $\Pi = \langle G(F) \cdot P_\varphi \rangle$.

In general

The assumption that π is compactly induced from $G(\mathcal{O}_F)$ allows us to choose the local groups at 0 and ∞ very simply. In general one shows they can be chosen to guarantee $P_\varphi(1) = \varphi(1) = 1$ by an argument on the Bruhat-Tits building of G.

The case of reductive G reduces easily to the semisimple case. We have not looked seriously at non-split G.

Theta series after all

As Wee Teck reminded me:

If X is an F-variety with G action, one may form the "X-theta series":

$$\theta_{X,f}(g) = \sum_{x \in X(F)} (g \cdot f)(x),\ f \in C_c(X(\mathbf{A})).$$

If $X = G$ with G acting by translation, θ_X is the Poincare series...

Wild ramification

Theorem

Suppose π is a pure supercuspidal compactly induced from an open compact subgroup that is **small** *in an appropriate sense. Then $\mathcal{L}(\pi)$ is wildly ramified.*

Arguing as before, one gets a non-vanishing Poincaré series that is unramified outside ∞, thus a local system on $\mathbb{A}^1$. By the previous theorem, $\mathcal{L}(\pi)$ is ramified, and since there are no tamely ramified local systems on $\mathbb{A}^1$, the ramification must be wild.
There are more general "small" open compacts – any pro-p open compact is "small" – but the general argument is more subtle.

Mixed supercuspidals

Start with any supercuspidal π and globalize to an automorphic Π such that $\mathcal{L}^{ss}(\Pi)$ is *irreducible* under the adjoint representation. For example, find a $k(Y)$ with points x, y, z and Π with $\Pi_x = \pi$ and $\Pi_y = Kl$ is the Kloosterman representation considered in Heinlöth-Ngô-Yun, for which the image of $\mathcal{L}(Kl)$ is known to be irreducible.

A result of Sawin-Templier generalizes purity of the MWF to $\hat{G}$-parameters: it implies that the Weil-Deligne parameter of the restriction to x of $\mathcal{L}^{ss}(\Pi)$ satisfies purity of MWF. In particular, the semisimple parameter $\mathcal{L}(\pi)$ has a completion $(\mathcal{L}(\pi), N)$ that satisfies purity of MWF, as required.

Lemma 2.5. *For $z \in \mathbb{G}_m(k)$ and $a = 1, \ldots, q-1$ we have*

$$(\alpha_a)^{(z)} = \beta^{az}.$$

Proof. Let K_z denote the completion of K at z, $\mathcal{O}_z$ its integer ring, k_z its residue field, which the inclusion of k in K canonically identifies with k. The image of the element $T \in \mathcal{O}_z$ in k_z is identified with the element $z \in k^{\times}$, so the residue field of the unramified extension $\mathcal{O}_z[T^{1/q-1}]$ is just $k(z^{1/q-1})$. The identity is then obvious. □

Lemma 2.6. *Let $\mathcal{L}$ be a rank one local system on $\mathbb{G}_{m,k}$. Then*

$$\mathrm{tr}(\mathrm{Frob}_z, \mathcal{L}) = \mathrm{tr}(\mathrm{Frob}_1, \mathcal{L})\,\mathrm{tr}(m_z^{-1}, \mathcal{L})$$

where m_z is the element of the inertia group I_0 which acts by multiplication by a on the $(q-1)$st root of T.

Application of potential automorphy

Start with an irreducible parameter $\rho : W_F \to \hat{G}(C)$. The methods of my paper with Böckle, Khare, and Thorne allow us to define a global Galois representation $\tau : Gal(K^{sep}/K) \to \hat{G}(C)$ for some $K = k(Y)$ with

- $\tau \mid_{\Gamma_x} \xrightarrow{\sim} \rho$ for some $x \in Y$;
- The image of τ is Zariski dense in $\hat{G}(C)$.
- For some finite Galois cover Y'/Y, $\tau' := \tau \mid_{Gal(K^{sep}/k(Y'))} = \mathcal{L}(\Pi')$ for some (cuspidal) automorphic representation of $G(\mathbf{A}_{k(Y')})$.

We aim to show that ρ is in the image of $\mathcal{L}$.

Assuming multiplicity one and stable base change

Assume there is stable cyclic base change for G over K. We can assume τ' is unramified everywhere. Now we want to descend (reverse of cyclic base change) step by step along a decomposition group $D_x \subset Gal(Y'/Y)$ over the point x.
Suppose Π' has mulitplicity one. Then we can at least descend over the center of D_x.
The multiplicity one hypothesis is unknown for general G but would follow from a similar hypothesis for the geometric Langlands correspondence over $\bar{k}(Y)$.

An inductive proof

We induct on $\dim G$, assume endoscopic transfer.
We also assume that the R-groups defined automorphically are isomorphic to the R-groups defined in terms of Galois parameters.
Finally, **we assume there are no pure incorrigible representations.**
Then all representations with pure unramified parameters are constituents of unramified principal series (known if $p \nmid |W(G)|$).
As in the proof for $GL(n)$, this should be the key step. Then it seems plausible that the strategy used by Arthur to construct tempered L-packets for classical groups can be applied inductively to this base change situation.