Dichotomy and Howe Duality for Exceptional Theta Correspondence

(Joint with Gordan Savin) 

Invent. math. 83, 229–255 (1986)

Inventiones mathematicae

On the local theta-correspondence*

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Introduction

Let $W, \langle\ ,\ \rangle$ be a non-degenerate symplectic vector space over a non-archimedian local field k of characteristic 0, and let $\tilde{S}p(W)$ be the non-trivial 2-fold central extension of the symplectic group $Sp(W)$. Fix a non-trivial additive character ψ of k and let (ω, S) be the corresponding (smooth) oscillator representation of $\tilde{S}p(W)$. If (G, G') is a reductive dual pair in $Sp(W)$, [7], and if $\tilde{G}$ and $\tilde{G}'$ are the inverse images of G and G' in $\tilde{S}p(W)$, then (ω, S) may be viewed as a representation of $\tilde{G}\times\tilde{G}'$. The local theta correspondence is defined as follows: if $\pi \in \operatorname{Irr}\tilde{G}$ is an irreducible smooth representation of $\tilde{G}$, let

$$\Theta(\pi; \tilde{G}') = \{\pi' \in \operatorname{Irr}\tilde{G}' \mid \operatorname{Hom}_{\tilde{G}\times\tilde{G}'}(\omega, \pi\otimes\pi') \neq 0\},$$

Author Citations for Stephen S. Kudla

Stephen S. Kudla is cited 1935 times by 541 authors

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Most Cited Publications	
Citations	**Publication**
122	**MR0818351 (87e:22037)** Kudla, Stephen S. On the local theta-correspondence. *Invent. Math.* 83 (1986), no. 2, 229–255. (Reviewer: Marie-France Vignéras) 22E50 (11F27 11F70)
117	**MR1289491 (95f:11036)** Kudla, Stephen S.; Rallis, Stephen A regularized Siegel-Weil formula: the first term identity. *Ann. of Math. (2)* 140 (1994), no. 1, 1–80. (Reviewer: Colette Mœglin) 11F70 (11F27 22E55)
96	**MR1286835 (95h:22019)** Kudla, Stephen S. Splitting metaplectic covers of dual reductive pairs. *Israel J. Math.* 87 (1994), no. 1-3, 361–401. (Reviewer: David Manderscheid) 22E50 (11F27 11F70 20G05)
87	**MR1327161 (96m:11041)** Harris, Michael; Kudla, Stephen S.; Sweet, William J. Theta dichotomy for unitary groups. *J. Amer. Math. Soc.* 9 (1996), no. 4, 941–1004. (Reviewer: Dipendra Prasad) 11F70 (11F27 22E50)
82	**MR1109355 (93a:11043)** Harris, Michael; Kudla, Stephen S. The central critical value of a triple product L-function. *Ann. of Math. (2)* 133 (1991), no. 3, 605–672. (Reviewer: Dipendra Prasad) 11F67 (11F27 11F41 11F70 22E55)
73	**MR1491448 (99j:11047)** Kudla, Stephen S. Central derivatives of Eisenstein series and height pairings. *Ann. of Math. (2)* 146 (1997), no. 3, 545–646. (Reviewer: Martin L. Karel) 11F46 (11F30 11G18 14G40)

Context of Kudla's Paper

Howe's theory of theta correspondence:

- V quadratic space over p-adic F, with isometry group $O(V)$
- W symplectic space over F, with isometry group $Sp(W)$
- Have a reductive dual pair

$$O(V) \times Sp(W) \longrightarrow Sp(V \otimes W)$$

- Ω a Weil representation of $O(V) \times Sp(W)$.

For $\pi \in \mathrm{Irr}(O(V))$, set

$$\Theta(\pi) = (\Omega \otimes \pi^\vee)_{O(V)} \quad \text{(big theta lift)}$$

This is a smooth representation of $Sp(W)$ such that $\pi \otimes \Theta(\pi)$ is the maximal π-isotypic quotient of Ω.

Howe Duality Conjecture: $\Theta(\pi)$ has a unique irreducible quotient $\theta(\pi)$. Moreover, $\theta(\pi_1) \cong \theta(\pi_2) \neq 0 \Longrightarrow \pi_1 \cong \pi_2$.

Main Results of Kudla's Paper

- $\Theta(\pi)$ has finite length, and thus has irreducible quotients.
- If π is supercuspidal, then $\Theta(\pi)$ is irreducible.
- (Tower property) Let W_n be the symplectic space of dimension $2n$ and $\Theta_n(\pi)$ the big theta lift of π to $\mathrm{Sp}(W_n)$. Then there exists n_0 such that

  $$\Theta_n(\pi) \neq 0 \iff n \geq n_0.$$

  This n_0 is called the first occurrence index of π in this tower of theta lifting. If π supercuspidal, then $\Theta_{n_0}(\pi)$ is supercuspidal and $\Theta_n(\pi)$ is noncuspidal for $n > n_0$.
- Compatibility of Θ with parabolic induction.

The introduction of the first occurrence index and his subsequent work with Rallis on the Siegel-Weil formula led them to formulate the local conservation relation.

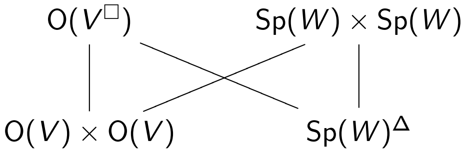
Main Ingredients in the Proof

- Jacquet modules of Weil representation. If $P = M \cdot N$ is a maximal parabolic of $\mathrm{Sp}(W)$, then Ω_N is a $M \times \mathrm{O}(V)$-module, which can be determined to a large extent (Kudla's filtration). This is useful for determining if $\Theta(\pi)$ is cuspidal.

$$\Omega \twoheadrightarrow \pi \otimes \Theta(\pi) \Longrightarrow \Omega_N \twoheadrightarrow \pi \otimes \Theta(\pi)_N.$$

  So $\Theta(\pi)$ is cuspidal if and only if π is not a quotient of Ω_N for all P.

- The doubling see-saw: $V^{\square} = V \oplus V^{-}$



Seesaw Pairs

Automorphic Forms of Several Variables
Taniguchi Symposium, Katata, 1983
Birkhäuser, 1984

SEESAW DUAL REDUCTIVE PAIRS

Stephen S. Kudla*

Introduction:

In this paper I want to discuss, in an informal style, a certain structure connected with the θ-correspondence for dual reductive pair [5], or, in more classical terminology, the theory of theta-functions. This structure, which I call a seesaw dual reductive pair or seesaw pair, gives rise to a certain family of identities between inner pro-ducts of automorphic forms on different groups. Identities of preci

Exceptional Dual Pairs

Goal of this talk: Discuss the analog of Kudla's results for some dual pairs in exceptional groups.

More precisely, consider the 3 dual pairs

$$G_2 \times \begin{cases} (\mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \subset E_6 \rtimes \mathbb{Z}/2\mathbb{Z} \\ PD^\times \subset E_6^D \\ \mathrm{PGSp}_6 \subset E_7 \end{cases}$$

where the exceptional groups of type E are all of adjoint type and D is a cubic division algebra.

This family of dual pairs is of the form:

$$\mathrm{Aut}(\mathbb{O}) \times \mathrm{Aut}(J) \subset E$$

where $\mathbb{O}$ is an octonion algebra and J is a Jordan algebra.

Minimal Representation

The role of the Weil representation is played by the minimal representation Ω of E. This is the smallest infinite-dimensional representation of E. Its local character expansion has the form

$$\Theta_\Omega \circ \exp = \widehat{\mu}_{\mathcal{O}_{min}} + c.$$

Alternatively, Ω can be realized as a space of functions on a relatively low-dimensional space (a quantization of $\mathcal{O}_{min}$). As illustration, we will describe a model of Ω of E_7 in the next few slides.

Unlike the Weil representation, one does not need to go to a nonlinear cover of E. So one does not have to deal with the intricacies of nonlinear cover.

Exceptional Jordan Algebra

Let J be the exceptional Jordan algebra of 3×3-Hermitian matrices with entries in $\mathbb{O}$. An element of J looks like:

$$X = \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix}$$

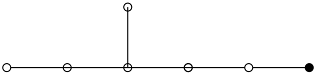
Some structures:

- $\det(X) = abc + Tr(xyz) - aN(x) - bN(y) - cN(z)$.
- $Tr(X) = a + b + c$
- Rank: say X has rank ≤ 1 if all its 2×2-minor are 0. The set $J_{\leq 1}$ of rank ≤ 1 elements is a cone in J with 0 as vertex.
- Jordan product: $X \circ Y = (XY + YX)/2$.

Siegel Parabolic of E_7

E_7 has an analog of the Siegel parabolic subgroup. This is a maximal parabolic subgroup $P = M \cdot N$ such that

- $N \cong J$ is abelian; we write an element of N as $n(X)$ for $X \in J$.
- $M = GE_6$ acts on $N = J$ by adjoint action, realizing M has the subgroup of $\mathrm{GL}(J)$ fixing $\det(X)$ up to scalars. Its derived group E_6 is the isometry group of det.



A Model for Ω

Recall that J_1 is the set of rank 1 elements in J. As P-modules:

$$C_c^\infty(J_1) \subset \Omega \subset C^\infty(J_1) \quad \text{with}$$

- $f \in \Omega$ vanishes at infinity.
- The asymptotics of elements of Ω at the vertex 0 is given by:

$$\Omega / C_c^\infty(J_1) \cong \Omega_N$$

- The action of $P = M \cdot N$ on $C^\infty(J_1)$ is given by:

$$(m \cdot f)(X) = \lambda(m) \cdot f(m^{-1} \cdot X)$$

$$(n(Y) \cdot f)(X) = \psi(Tr_J(Y \circ X)) \cdot f(X).$$

It remains to specify the action of a Weyl group element w. This acts by a "Fourier transform on the cone J_1". Such models allow one to compute the Jacquet modules of Ω for different parabolic subgroups of $G_2 \times H$, just as Kudla did in the classical case.

Big Theta Lifts

Now for $\pi \in \mathrm{Irr}(G_2)$, we may consider

$$\Theta(\pi) := (\Omega \otimes \pi^\vee)_{G_2}$$

which is a smooth representation of H such that $\pi \otimes \Theta(\pi)$ is the maximal π-isotypic quotient of Ω.

Questions:

(a) Is $\Theta(\pi)$ of finite length?

(b) Does $\Theta(\pi)$ (if nonzero) have irreducible quotients?

(c) Does $\Theta(\pi)$ (if nonzero) have a unique irreducible quotient?

(d) Does the analog of Howe duality hold?

These questions all concern upper bounds for $\Theta(\pi)$.

The Theorems

Theorem (Howe Duality)

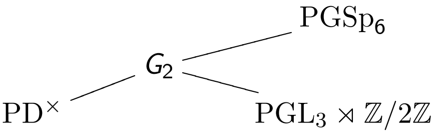
Consider any one of the 3 dual pairs above.

(i) For $\pi \in \mathrm{Irr}(G_2)$, $\Theta(\pi)$ has finite length and a unique irreducible quotient $\theta(\pi)$ (if nonzero).

(ii) $\theta(\pi_1) \cong \theta(\pi_2) \neq 0 \implies \pi_1 \cong \pi_2$.

Theorem (Dichotomy)

Let $\pi \in \mathrm{Irr}(G_2)$. Then π has nonzero theta lift to exactly one of $PD^\times$ or $PGSp_6$.



Finiteness and Irreducible Quotients

Now consider these questions:

(a) Is $\Theta(\pi)$ of finite length?

(b) Does $\Theta(\pi)$ (if nonzero) have irreducible quotients?

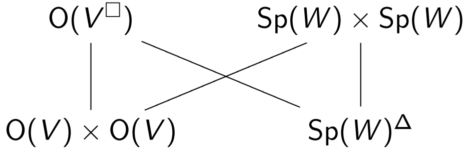
Clearly (a) implies (b). But in practice, we first show (b) before showing (a). For $\pi \in \mathrm{Irr}(G_2)$, write

$$\Theta(\pi) = \Theta(\pi)_c \oplus \Theta(\pi)_{nc}$$

If $\Theta(\pi)_c \neq 0$, then it certainly has irreducible quotients. We show:

- $\Theta(\pi)_{nc}$ has finite length (and thus has irreducible quotients, completing proof of (b)). This is achieved via Jacquet module computations (and induction on size of groups).
- $\Theta(\pi)_c$ is irreducible or zero (thus completing proof of (a)). This is done in the course of proving Howe duality, and is where the analog of the doubling see-saw is needed.

Interlude: Doubling See-Saw Argument



- Seesaw identity: for $\pi, \pi' \in \mathrm{Irr}(O(V))$,

  $$\mathrm{Hom}_{Sp(W)}(\Theta(\pi') \otimes \Theta(\pi^\vee), \mathbb{C}) \cong \mathrm{Hom}_{O(V) \times O(V)}(\Theta(1), \pi' \otimes \pi^\vee)$$

- Local Siegel-Weil (Rallis),

  $$\Theta(1) \subset I(s_0) \quad \text{a Siegel degenerate principal series.}$$

- Mackey restriction: as a $O(V) \times O(V)$-module

  $$I(s_0) \supset C_c^\infty(O(V)) \quad \text{with small quotient.}$$

So for supercuspidal π, π',

$$\mathrm{Hom}_{Sp(W)}(\theta(\pi'), \theta(\pi)) \subset \mathrm{Hom}_{O(V)^2}(C_c^\infty(O(V)), \pi' \otimes \pi^\vee)$$

A General Principle

There is no doubling seesaw in exceptional Θ-correspondence.

However, the doubling seesaw argument is an instance of:

General principle: For a dual pair $G \times H$, theta correspondence often relates a period $\mathcal{P}$ on G to a period $\mathcal{Q}$ on H.

A period $\mathcal{P}$ on G is given by a pair (G', χ) where $G' \subset G$ and $\chi : G' \to \mathbb{C}^\times$. For $\pi \in \mathrm{Irr}(G)$, the $\mathcal{P}$-period of π refers to $\mathrm{Hom}_{G'}(\pi, \chi)$.

Explication: For $\pi \in \mathrm{Irr}(G)$ and a period $\mathcal{Q}$ on H, there is a corresponding period $\mathcal{P}$ on G such that

$$\mathcal{P}(\pi) := \mathcal{P}\text{-period of } \pi \cong \mathcal{Q}\text{-period of } \Theta(\pi) =: \mathcal{Q}(\Theta(\pi)).$$

Period Ping Pong

We shall explain how the above principle can be exploited to prove Howe duality. Start with:

- $G \times H$ is a dual pair
- $\pi \in \mathrm{Irr}(G)$ and $\sigma \in \mathrm{Irr}(H)$ such that $\Omega \twoheadrightarrow \pi \otimes \sigma$, i.e.

$$\Theta(\pi) \twoheadrightarrow \sigma \quad \text{and} \quad \Theta(\sigma) \twoheadrightarrow \pi.$$

- a (seed) period $\mathcal{Q}_1$ on H (e.g. Whittaker period).

Applying the principle, there is a period $\mathcal{P}_1$ on G such that

$$\mathcal{P}_1(\pi) \cong \mathcal{Q}_1(\Theta(\pi)) \supset \mathcal{Q}_1(\sigma).$$

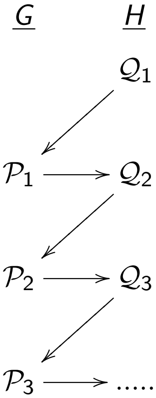
Applying the principle again (but the other way round), there is a period $\mathcal{Q}_2$ on H such that

$$\mathcal{Q}_2(\sigma) \cong \mathcal{P}_1(\Theta(\sigma)) \supset \mathcal{P}_1(\pi).$$

Inductively, get a sequence of periods $\mathcal{P}_i$ on G and $\mathcal{Q}_i$ on H with

$$\mathcal{Q}_{i+1}(\sigma) \cong \mathcal{P}_i(\Theta(\sigma)) \supset \mathcal{P}_i(\pi) \cong \mathcal{Q}_i(\Theta(\pi)) \supset \mathcal{Q}_i(\sigma)$$

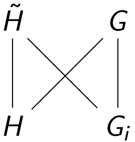
Period Ping Pong II



Question: How does this rally end?

A See-Saw Battle

Empirically, the subgroups G_i associated to $\mathcal{P}_i$ becomes increasingly reductive. If G_i is reductive, the principle takes a modified form. Indeed, the computation of $\mathcal{P}_i(\sigma)$ typically involves a seesaw diagram:



The seesaw identity gives:

$$\mathcal{P}_i(\Theta(\sigma)) = \mathrm{Hom}_{G_i}(\Theta(\sigma), 1) \cong \mathrm{Hom}_H(\Theta(1), \sigma).$$

So we end up with a coperiod of σ on H, instead of a period.

Siegel-Weil and Mackey

Local Siegel-Weil: $\Theta(1) \subset I(s_0)$, a degenerate principal series of $\tilde{H}$. Then

$$\mathrm{rest} : \mathrm{Hom}_H(I(s_0), \sigma) \longrightarrow \mathrm{Hom}_H(\Theta(1), \sigma) = \mathcal{P}_i(\Theta(\sigma)).$$

We are led to understand $I(s_0)$ as a H-module, via Mackey theory.

Miracle: as a H-module,

$$I(s_0) \supset ind_{H_1}^{H} \chi_1 \quad \text{with small quotient}$$

Hence

$$\mathcal{P}_i(\Theta(\sigma)) \longleftarrow \mathrm{Hom}_H(I(s_0), \sigma) \longrightarrow \mathcal{Q}_1(\sigma).$$

Key: For almost all σ, the arrow on the left is surjective and the arrow on the right is an isomoprhism, so that

$$\mathcal{Q}_1(\sigma) \supset \mathcal{P}_i(\Theta(\sigma)) \supset \mathcal{P}_i(\pi).$$

Merry Go Round

Hence we have:

$$\mathcal{Q}_1(\sigma) \supset \mathcal{P}_i(\Theta(\sigma)) \supset \mathcal{P}_i(\pi) \cong .... \supset \mathcal{P}_1(\pi) \supset \mathcal{Q}_1(\Theta(\pi)) \supset \mathcal{Q}_1(\sigma)$$

If one of these spaces is finite-dimensional, then equality holds throughout!

How does this help towards Howe Duality? Suppose the above spaces have finite dimension $d \neq 0$:

$$0 \neq d = \dim \mathcal{Q}_1(\sigma) = \dim \mathcal{Q}_1(\Theta(\pi)) = \dim \mathcal{P}_1(\pi) = \dim \mathcal{P}_1(\Theta(\sigma)).$$

Now if $\Theta(\pi) \twoheadrightarrow \sigma \oplus \sigma'$, the above equality holds with σ and σ', leading to the contradiction:

$$d = \dim \mathcal{Q}_1(\Theta(\pi)) \geq \dim \mathcal{Q}_1(\sigma) + \dim \mathcal{Q}_1(\sigma') = 2d.$$

The Periods We Use

For our dual pair $G_2 \times H$ with $H = \mathrm{Aut}(J)$, we play this period ping pong 2 times:

- Whittaker on $G_2 \to$ Shalika on $\mathrm{PGSp}_6 \to$ Fourier-Jacobi on G_2
- Fourier coefficients of G_2 wrt Heisenberg parabolic $P = MN$.

The generic characters of N are parametrized by étale cubic F-algebras E. For any given E, consider the (N, ψ_E)-period:

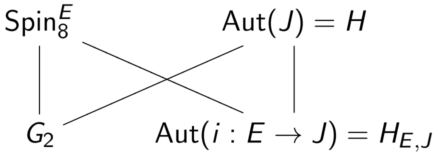
$$\Omega_{N,\psi_E} \cong \mathrm{ind}_{H_{E,J}}^{H}(1), \quad \text{with } H_{E,J} = \mathrm{Aut}(i : E \hookrightarrow J)$$

Thus, (N, ψ_E)-period on $G_2 \to H_{E,J}$-period on $\mathrm{Aut}(J)$, with

$$H_{E,J} = \begin{cases} PE^{\times} \text{ if } H = \mathrm{PGL}_3 \text{ or } PD^{\times}; \\ SL_2(E)/\mu_2 \text{ if } H = \mathrm{PGSp}_6 \end{cases}$$

A See-Saw diagram

As $H_{E,J}$ is reductive, the next step is a seesaw:



- Local SW: Theta lift of the trivial representation of $H_{E,J}$ is a subquotient of a degenerate p.s. $I(1/2)$ of Spin_8^E (induced from Heisenberg parabolic).
- Mackey: As a G_2-module:

$$I(1/2) \supset ind_N^{G_2} \psi_E \quad \text{with small quotient}$$

A Cycle of Containments

This gives the following chains of containments, for $\pi \in \mathrm{Irr}(G_2)$ and $\tau \in \mathrm{Irr}(H)$ such that $\Omega \twoheadrightarrow \pi \otimes \tau$:

$$\mathrm{Hom}_N(\pi, \psi_E) \subset \mathrm{Hom}_N(\Theta(\tau), \psi_E) \cong \mathrm{Hom}_{H_{E,J}}(\tau^\vee, 1)$$

and

$$\mathrm{Hom}_{H_{E,J}}(\tau^\vee, 1) \subset \mathrm{Hom}_{H_{E,J}}(\Theta(\pi^\vee), 1) \cong \mathrm{Hom}_N(\pi, \psi_E),$$

where the last isomorphism holds for all tempered π.

When π is nongeneric, $\dim \mathrm{Hom}_N(\pi, \psi_E)$ is finite for all E and is nonzero for some E. In this case, equality holds throughout.

A Theta Book?

Lecture Notes on Local Theta Correspondence

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HAPPY BIRTHDAY, STEVE!