CM values of higher automorphic Green functions

(joint work with S. Ehlen and T. Yang)

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The automorphic Green function

- Throughout: $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{H}$ = complex upper half-plane.
- Consider the modular curve $X = \Gamma\backslash\mathbb{H}$.
- The automorphic Green function on $X \times X$ is defined for $\mathrm{Re}(s) > 1$ as

$$G_s(z_1, z_2) = -2 \sum_{\gamma\in\Gamma} Q_{s-1}\left(1 + \frac{|z_1 - \gamma z_2|^2}{2\,\mathrm{Im}(z_1)\,\mathrm{Im}(\gamma z_2)}\right),$$

  where $Q_{s-1}(t)$ is the Legendre function $Q_{s-1}(t) = \int_0^\infty (t + \sqrt{t^2-1}\cosh(u))^{-s}\,du$.
- It plays an important role in the analytic and arithmetic theory of automorphic forms.

Properties

The function $G_s(z_1, z_2)$ has the properties:

- Γ-invariant in both variables
- is smooth on $(X \times X) \setminus$ diagonal
- $G_s(z_1, z_2) = |\bar{\Gamma}_{z_2}| \cdot \log |z_1 - z_2|^2 +$ continuous, as $z_1 \to z_2$
- $\Delta_i G_s = s(1-s) G_s$, where

$$\Delta_i = -y_i^2 \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right).$$

- $G_s(z_1, z_2) = O(y_1^{1-s})$ as $y_1 \to \infty$

Proposition

These properties characterize $G_s(z_1, z_2)$ uniquely.

CM points

A point $z \in \mathbb{H}$ is called a CM point if

$$az^2 + bz + c = 0$$

for integers $a, b, c \in \mathbb{Z}$ with $a > 0$ and $d = b^2 - 4ac < 0$.

- Γ acts on the set $\mathrm{CM}(d)$ of CM points of fixed discriminant d.
- Get the special divisor $Z(d) = \Gamma \backslash \mathrm{CM}(d)$ on X.
- Here points are counted with multiplicity $1/|\overline{\Gamma}_z|$.

The algebraicity conjecture

- Let $m \in \mathbb{Z}_{>0}$, and let T_m be the m-th Hecke operator. Put
$$G_s^m(z_1, z_2) = G_s(z_1, z_2) \mid T_m.$$
- For $k \in \mathbb{Z}_{>0}$ and $f = \sum_n c_f(n) q^n \in M^!_{2-2k}$ with integral coefficients $c_f(n)$, consider the linear combination
$$G_{k,f}(z_1, z_2) = \sum_{m>0} c_f(-m) m^{k-1} G_k^m(z_1, z_2)$$

Conjecture (Gross-Zagier)

Let $z_1, z_2 \in \mathbb{H}$ be CM points of discriminants $d_1, d_2 < 0$ such that (z_1, z_2) is not contained in the divisor of $G_{k,f}$. Then
$$(d_1 d_2)^{\frac{k-1}{2}} G_{k,f}(z_1, z_2) = \log|\alpha|$$
for an $\alpha \in H_{d_1} \cdot H_{d_2}$, where H_d is the ring class field of the order $\mathcal{O}_d \subset \mathbb{Q}(\sqrt{d})$ of discriminant d.

Previous results

- $k = 1$: $G_1(z_1, z_2)$ "=" $\log |j(z_1) - j(z_2)|^2$, so in this case the conjecture is a theorem by CM theory.
- Gross-Kohnen-Zagier: numerical evidence and formulas for averages $G_{k,f}(Z(d_1), Z(d_2))$ for odd k.
- S. Zhang: conditional result for $\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})$.
- A. Mellit: $k = 2$, $z_2 = i$.
- M. Viazovska: k arbitrary, $\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})$.
- Y. Li: averages for even k and prime factorization.

Our results: partial averages

Theorem (B.-Ehlen-Yang)

Let $k > 1$ and $f \in M^!_{2-2k}$ with integral coefficients. Let $d_1, d_2 < 0$ be discriminants, d_1 fundamental, $d_1 d_2 \neq \square$. Fix a CM point z_2 of discriminant d_2. Then

$$(d_1 d_2)^{\frac{k-1}{2}} G_{k,f}(Z(d_1), {z_2}^\sigma) = \frac{1}{r} \cdot \log |\alpha^\sigma|,$$

for an $\alpha \in H$ and $r \in \mathbb{Z}_{>0}$, and for all $\sigma \in \mathrm{Gal}(H/K)$. Here for k odd: $K = \mathbb{Q}(\sqrt{d_2})$ and $H = H_{d_2}$; and for k even: $K = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ and $H = H_{d_2}(\sqrt{d_1})$.

- Hence, the conjecture is true for $\mathbb{Q}(\sqrt{d_1})$ of class number one. (More generally when the class group has exponent two.)
- Very recently, Y. Li was able to remove the partial average and prove it in general.

Approach

- This result is a consequence of a general explicit formula for CM values of higher automorphic Green functions on othogonal Shimura varieties.
- Also obtain higher dimensional analogues.
- Key ingrdients: Regularized theta correpondence (Borcherds, B.), see-saw dual pairs (Kudla), Rankin-Cohen-operators, harmonic Maass forms of weight 1 (Duke-Li, Ehlen).

Orthogonal Shimura varieties

- (V, Q) quadratic space over $\mathbb{Q}$ of signature $(n, 2)$.
- $H = \mathrm{GSpin}(V)$ (an extension of $\mathrm{SO}(V)$).
- $\mathbb{D} = \{z \subset V(\mathbb{R});\ \dim(z) = 2,\ Q\,|_z < 0, \text{ oriented}\}$.
- $K \subset H(\mathbb{A}_f)$ compact open.
- We consider
$$X_K = H(\mathbb{Q})\backslash\mathbb{D} \times H(\mathbb{A}_f)/K.$$
- Generalizes modular curves ($n = 1$) and Hilbert modular surfaces ($n = 2$).

Automorphic Green functions

- Let $L \subset V$ be an even lattice. Assume K stabilzes $\hat{L}$ and acts trivially on L'/L.
- For $\mu \in L'/L$ and $m \in \mathbb{Z} + Q(\mu)$ positive there is a special divisor $Z(m, \mu)$ along X_K.

The automorphic Green function on X_K for $Z(m, \mu)$ is defined by:

$$\Phi_{m,\mu}(z, h, s)$$

$$\doteq \sum_{\substack{\lambda \in h(\mu+L) \\ Q(\lambda)=m}} \left(\frac{m}{Q(\lambda_{z^\perp})}\right)^{s+\frac{n}{4}-\frac{1}{2}} F\left(s+\frac{n}{4}-\frac{1}{2}, s-\frac{n}{4}+\frac{1}{2}, 2s; \frac{m}{Q(\lambda_{z^\perp})}\right)$$

- Converges for $\mathrm{Re}(s) > s_0 := \frac{n}{4} + \frac{1}{2}$.
- Logarithmic singularity on $Z(m, \mu)$.

Examples related to modular curves

- For $L = M_2(\mathbb{Z})$ (signature $(2,2)$), we obtain

$$\Phi_m(z,s) = -\frac{2}{\Gamma(s)} G_s(z_1, z_2) \mid T_m$$

  with $z = (z_1, z_2) \in \mathbb{H}^2$.
- For $L = M_2^0(\mathbb{Z})$ (signature $(1,2)$), we obtain

$$\Phi_m(z,s) = -\frac{2}{\Gamma(s + \frac{1}{4})} G_{2s-\frac{1}{2}}(Z(m), z),$$

  where $z \in \mathbb{H}$.

Special values of the spectral parameter

In analogy with the case of modular curves look at special spectral parameter $s_0 + j$ where $s_0 := \frac{n}{4} + \frac{1}{2}$ and $j \in \mathbb{Z}_{\geq 0}$. Let $\kappa = 1 - \frac{n}{2}$.

- For $f \in M^!_{\kappa-2j,\bar{\rho}_L}$ with coefficients $c_f(m, \mu)$ define

$$\Phi^j(z, h, f) \doteq \sum_{m>0,\mu} c_f(-m, \mu) m^j \Phi_{m,\mu}(z, h, s_0 + j).$$

Proposition

For $f \in M^!_{\kappa-2j,\bar{\rho}_L}$ the function $\Phi^j(z, h, f)$ is equal to the theta lift

$$\Phi^j(z, f, h) = \frac{1}{(4\pi)^j} \int^{reg}_{\Gamma\backslash\mathbb{H}} (R^j_{\kappa-2j} f)(\tau) \cdot \theta_L(\tau, z, h)\, d\mu(\tau),$$

where $\theta_L(\tau, z, h)$ is the Siegel theta function of L, and R^j_ℓ is the (iterated) Maass raising operator.

CM points

Let $U \subset V$ be *rational* 2-dimensional negative subspace.

- Obtain two points $z_U^\pm = U_\mathbb{R}^\pm \in \mathbb{D}$.
- $T = \mathrm{GSpin}(U)$ determines a torus in $H = \mathrm{GSpin}(V)$.
- Get a CM cycle

$$Z(U) = T(\mathbb{Q})\backslash(\{z_U^\pm\} \times T(\mathbb{A}_f)/K_T) \longrightarrow X_K.$$

- Also get *definite* even lattices

$$N = L \cap U, \qquad P = L \cap U^\perp,$$

and theta functions θ_N and θ_P of weights -1 and $n/2$.

CM values of higher Green functions

For $h \in T(\mathbb{A}_f)$, fix a harmonic Maass form $\mathcal{G}_{N,h}$ with $L_1 \mathcal{G}_{N,h} = \theta_N(\tau, h)$. Write $\mathcal{G}_{N,h}^+$ for its holomorphic part.

Theorem (B.-Ehlen-Yang)

Let $f \in M^!_{\kappa-2j,\bar{\rho}_L}$. Then we have

$$\Phi^j(z_U^\pm, h, f) = \mathrm{CT}\left(f \cdot [\theta_P, \mathcal{G}_{N,h}^+]_j\right).$$

Here $\mathrm{CT}(\cdot)$ denotes the constant term of a q-series, and $[\theta_P, \mathcal{G}_{N,h}^+]_j$ is the j-th Rankin-Cohen bracket.

- If f has integral coefficients, then the CM value is an explicit integral linear combination of the coefficients of $\mathcal{G}_{N,h}^+$.

Harmonic Maass forms of weight one

Theorem

There exists a $\mathcal{G}_{N,h} \in H_{1,\rho_N}$ *with* $L_1(\mathcal{G}_{N,h}) = \theta_N(\tau, h)$ *and holomorphic part*

$$\mathcal{G}^+_{N,h}(\tau) = \sum_{n \gg -\infty} c^+_{N,h}(n) q^n,$$

such that

$$c^+_{N,h}(n) = \frac{1}{r} \log |\alpha_{N,h}(n)|,$$

where $\alpha_{N,h}(n) \in H_D$ *with* $D = \operatorname{disc}(N) < 0$ *and* $r \in \mathbb{Z}_{>0}$ *is independent of* n *and* h*.*

Remark

- *There is an algorithm to compute the* $\alpha_{N,h}(n)$*.*
- *Generalizes of previous results by Duke-Li and Ehlen.*

Harmonic Maass forms of weight one

Combining the last two results we obtain:

Corollary

Let $f \in M^!_{\kappa-2j,\bar{\rho}_L}$ with integral Fourier coefficients. Then

$$\Phi^j(z_U^\pm, h, f) = \mathrm{CT}\left(f \cdot [\theta_P, \mathcal{G}^+_{N,h}]_j\right) = \frac{1}{r}\log|\alpha_{N,h}|,$$

where $\alpha_{N,h} \in H_D$ with $D = \operatorname{disc} N$, and $r \in \mathbb{Z}_{>0}$ independent of f.

Averages over CM cycles

Motivated by work of J. Schofer and Steve.

- In the CM value formula for $\Phi^j(z_U^+, h, f)$ average over $Z(U)$.
- The average of the $\mathcal{G}_{N,h}$ is the central derivative $\mathcal{E}_N(\tau) = E_N'(\tau, 0; 1)$ of the incoherent Eisenstein series of weight 1 for N. See the IMRN paper of KRY.

Theorem (B.-Ehlen-Yang)

Let $f \in H_{\kappa-2j,\bar{\rho}_L}$. Then we have

$$\frac{1}{\deg(Z(U))}\Phi^j(Z(U), f) = \mathrm{CT}\left(f^+ \cdot [\theta_P, \mathcal{E}_N^+]_j\right) - L'\left(\xi_{\kappa-2j}(f), U, 0\right).$$

Here $L(g, U, s)$ is a certain convolution L-function of a cusp form $g \in S_{1+n/2+2j,\rho_L}$ and the theta series θ_P of weight $n/2$.

Specializing to $O(1,2)$

- Take $L = M_2^0(\mathbb{Z})$ of signature $(1,2)$.
- $s_0 = \frac{n}{4} + \frac{1}{2} = \frac{3}{4}$.
- Thus, $\Phi^j(z, f_m) \doteq G_{2(s_0+j)-\frac{1}{2}}(Z(m), z) = G_{1+2j}(Z(m), z)$.
- So we obtain G_k for k odd.
- Variant: replace θ_L by a theta function with a degree 1 polynomial. This way we get $G_k(Z(m), z)$ for all $k \in \mathbb{Z}_{>0}$.

Specializing to O(1, 2)

For a fundamental discriminant d_1, let
$\mathrm{Za}_{d_1} : M^!_{2-2k} \to M^!_{3/2-k}(\bar{\rho}_L)$ be the d_1-th Zagier lift.

Theorem

Let $f \in M^!_{2-2k}$ with integral coefficients.

- *If $k > 1$ is odd, then*

$$G_{k,f}(Z(d_1), z_U^+) = -2^{k-2}\ \mathrm{CT}\left(\mathrm{Za}_{d_1}(f) \cdot [\theta_P, \mathcal{G}_N^+]_{\frac{k-1}{2}}\right).$$

- *If $k > 1$ is even, there is an analogous formula involving a weight 3/2 unary theta function and some twisting.*

- Corollary: This quantity is of the form $\frac{1}{r}\log|\alpha_N|$ with $\alpha_N \in H_{d_2}$ for $d_2 = \mathrm{disc}(N)$, and $r \in \mathbb{Z}_{>0}$.

An example

- Numerically, $G_3(i, \frac{1+i\sqrt{23}}{2}) \approx -1.000394556341$.
  Here $k = 3$, $d_1 = -4$, $d_2 = -23$.
- In this case, $N \cong \mathcal{O}_{-23}$, the ring of integers in $\mathbb{Q}(\sqrt{-23})$ with $-\mathrm{N}_{\mathbb{Q}(\sqrt{-23})/\mathbb{Q}}$ as quadratic form.
- Let $f = E_4^2/\Delta = q^{-1} + O(1) \in M^!_{-4}$.
  Then $G_3 = G_{3,f}$.
- Let $\mathcal{G}_N \in H_{1,\rho_N}$ be "nice" with $L_1(\mathcal{G}_N) = \theta_N$.
- Our formula implies that

$$G_3\left(i, \frac{1+i\sqrt{23}}{2}\right) = -\frac{25}{23}c_N^+(7) - \frac{4}{23}c_N^+(14) + \frac{11}{23}c_N^+(19) + \frac{20}{23}c_N^+(22) + \frac{1}{2}c_N^+(23) + \frac{378}{23}c_N^+(-1).$$

An example

- Let H_{-23} be the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$.
- Let $\alpha \approx 1.324717957244$ be the unique real root of $x^3 - x - 1$.
- Then α is a generator of H_{-23} over $\mathbb{Q}(\sqrt{-23})$ and also a generator of the unit group of the real subfield $\mathbb{Q}(\alpha) \subset H_{-23}$.

Using the results on harmonic Maass forms of weight one we obtain

$$G_3\left(i, \frac{1+i\sqrt{23}}{2}\right)$$
$$= \frac{1}{23}\log\left|\alpha^{294} \cdot \frac{(\alpha^2-2\alpha-1)^{50}\,(3\alpha^2-5\alpha+1)^8}{(4\alpha^2-\alpha+2)^{40}\,(\alpha^2-4\alpha+3)^{22}\,(-3\alpha^2+2\alpha+1)^{23}}\right|.$$

Application: Higher weight Gross-Kohnen-Zagier

Consider higher Heegner cycles $Z_k(d)$ over $X_0(M)$.

Corollary

Let $k \geq 1$ be odd and $D < 0$ be a fixed fundamental discriminant, coprime to $2M$. The height pairing generating series

$$A_{k,D}(\tau) = \sum_{(d,D)=1} \langle Z_k(d), Z_k(D) \rangle \cdot q^{|d|}$$

is a cusp form of weight $k + 1/2$.

Happy birthday Steve!