Sup-norm bounds for Siegel modular forms

Jürg Kramer

Work in progress jointly with Antareep Mandal

Department of Mathematics
Humboldt-University Berlin

6 July 2021

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Strategy of proof

Heat kernel for Γ

Concluding the proof

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Some notation

- $\mathbb{H} := \{z \in \mathbb{C} \,|\, z = x + iy,\ y > 0\}$, upper half-plane.
- $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, Fuchsian subgroup of first kind.
- $\mathcal{F}_\Gamma \subset \mathbb{H}$, fundamental domain for Γ.
- $S_2(\Gamma)$ $\mathbb{C}$-vector space of cusp forms of weight 2 for Γ.
- $\{f_1, \ldots, f_d\}$ orthonormal basis of $S_2(\Gamma)$.
- $B_2^\Gamma(z) := \sum_{j=1}^{d} |f_j(z)|^2 y^2$.

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Theorem (with J. Jorgenson, 2004). With the above notations we have

$$\sup_{z \in \mathcal{F}_\Gamma} B_2^\Gamma(z) = O_{\Gamma_0}(1),$$

where $\Gamma \subseteq \Gamma_0$ and $\Gamma_0 \subset \mathrm{PSL}_2(\mathbb{R})$ is a fixed Fuchsian subgroup.

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Significance of the result

On $\mathbb{H}$, there are two fundamental metrics represented by the following two $(1,1)$-forms:

$$\mu_{\mathrm{hyp}}(z) := \frac{i}{2}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{y^2}, \text{ the hyperbolic metric;}$$

$$\mu_{\mathrm{can}}(z) := \frac{1}{g_\Gamma}\frac{i}{2}\sum_{j=1}^{d}|f_j(z)|^2\mathrm{d}z \wedge \mathrm{d}\bar{z}, \text{ the canonical metric.}$$

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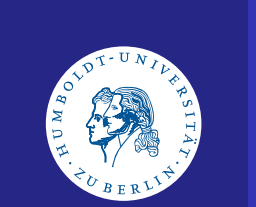
The comparison of the two $(1,1)$-forms yields ($g_\Gamma = \text{genus}$)

$$\frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)} = \frac{1}{g_\Gamma}B_2^\Gamma(z).$$

This enables applications of Arakelov theory to modular curves: E.g., in order to give bounds for the height or Faltings's delta of modular curves, one needs bounds for the ratio $\mu_{\mathrm{can}}/\mu_{\mathrm{hyp}}$.



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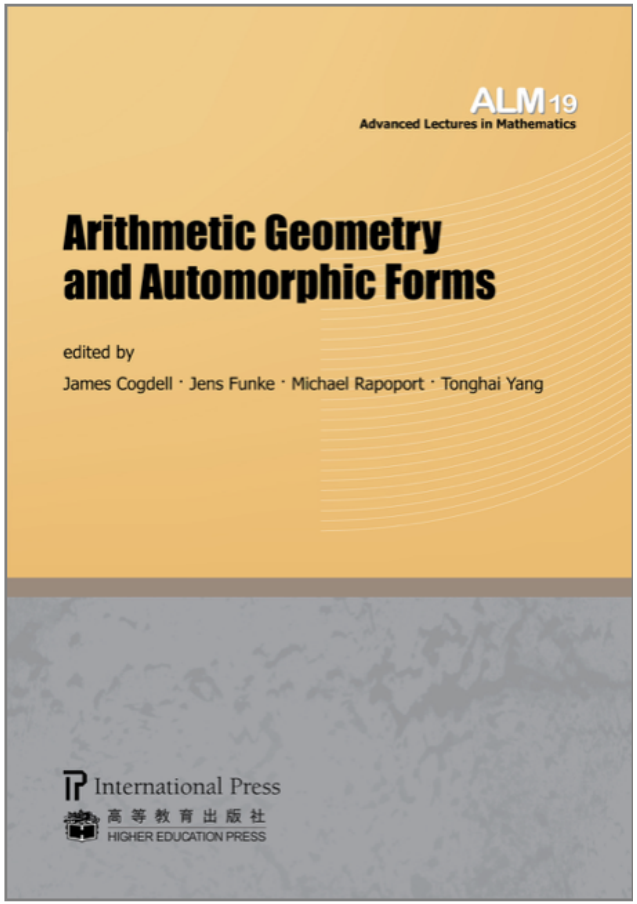

Our contribution to Steve's 60th birthday:

"Sup-norm bounds for automorphic forms and Eisenstein series"

There, we gave a proof of the optimal sup-norm bounds for $k = 2$ using (parabolic), hyperbolic, and elliptic Eisenstein series à la Kudla–Millson.

Sup-norm bounds in the elliptic modular case

Generalization to arbitrary even weight k

- $S_k(\Gamma)$ $\mathbb{C}$-vector space of cusp forms of weight k for Γ.
- $\{f_1, \ldots, f_d\}$ orthonormal basis of $S_k(\Gamma)$.
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Theorem (with J. Friedman and J. Jorgenson, 2016).
If the subgroup Γ is cocompact, then we have

$$\sup_{z \in \mathcal{F}_\Gamma} B_k^\Gamma(z) = O_\Gamma(k).$$

If the subgroup is cofinite, but not cocompact, then we have

$$\sup_{z \in \mathcal{F}_\Gamma} B_k^\Gamma(z) = O_\Gamma(k^{3/2}).$$

Ideas of proof

Ideas of proof

Some more notation

- Let $\mathcal{V}_k(\Gamma)$ denote the space of functions $\varphi\colon \mathbb{H} \longrightarrow \mathbb{C}$ satisfying

$$\varphi\left(\frac{az+b}{cz+d}\right)\left(\frac{cz+d}{c\overline{z}+d}\right)^{-k/2} = \varphi(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

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- $\mathcal{H}_k(\Gamma) := \left\{\varphi \in \mathcal{V}_k(\Gamma) \,\middle|\, \|\varphi\| < \infty\right\}$, where

$$\|\varphi\| := \int_{\mathcal{F}_\Gamma} |\varphi(z)|^2 \, \mu_{\mathrm{hyp}}(z),$$

  which becomes a Hilbert space.

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- $\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$, hyperbolic Laplacian of weight k, acts on the dense subspace of smooth functions.

Ideas of proof

Consequences

- If $\varphi \in \mathcal{H}_k(\Gamma)$ is an eigenfunction of Δ_k, hence a Maass form of weight $k/2$ for Γ, i.e., $\Delta_k \varphi(z) = \lambda\, \varphi(z)$, then we have $\lambda \geq k(2-k)/4$.

Ideas of proof

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- Furthermore, we find the isomorphism

$$S_k(\Gamma) \cong \ker\big(\Delta_k - k(2-k)/4\big),$$

which is induced from the assignment $f(z) \mapsto f(z)y^{k/2}$.

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  which is induced from the assignment $f(z) \mapsto f(z)y^{k/2}$.
- This shows that $B_k^\Gamma(z)$ can be seen as part of the spectral expansion of the heat kernel $K_k^\Gamma(t;z,w)$ of weight k for Γ on the diagonal; here $t \in \mathbb{R}_{>0}$ and $z, w \in \mathbb{H}$.
- More precisely, for all $t \in \mathbb{R}_{>0}$, we get the bound

$$e^{-k(2-k)t/4}B_k^\Gamma(z) \leq K_k^\Gamma(t;z,z).$$

Ideas of proof

Heat kernel of weight k on $\mathbb{H}$

For $t \in \mathbb{R}_{>0}$ and $r \in \mathbb{R}_{>0}$, we have

$$K_k(t;r) = \frac{\sqrt{2}\, e^{t/4}}{(4\pi t)^{3/2}} \int_r^\infty \frac{\varrho\, e^{-\varrho^2/(4t)}}{\sqrt{\cosh(\varrho) - \cosh(r)}} \underbrace{T_k\left(\frac{\cosh(\varrho/2)}{\cosh(r/2)}\right)}_{k\text{-th Chebyshev pol.}} \mathrm{d}\varrho.$$

Ideas of proof

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Heat kernel of weight k for Γ

For $t \in \mathbb{R}_{>0}$ and $z, w \in \mathbb{H}$ with $r = r_{z,\gamma w} = \mathrm{dist}_{\mathrm{hyp}}(z, \gamma w)$ $(\gamma \in \Gamma)$, we then obtain

$$K_k^\Gamma(t;z,w) = \sum_{\gamma \in \Gamma} \left(\frac{c\overline{w}+d}{cw+d}\right)^{k/2} \left(\frac{z-\gamma\overline{w}}{\gamma w - \overline{z}}\right)^{k/2} K_k(t; r_{z,\gamma w}).$$

Ideas of proof

Remaining technical steps

- It is crucial to get a bound for the k-th Chebyshev polynomial:

$$T_k\left(\frac{\cosh(\varrho/2)}{\cosh(r/2)}\right) \leq \frac{e^{k\varrho/2}}{\cosh^k(r/2)}. \qquad (*)$$

Ideas of proof

Remaining technical steps

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- Γ cocompact: Using the bound $(*)$, the claimed bound $O_\Gamma(k)$ follows from bounding $K_k^\Gamma(t;z,w)$ for $z=w$, using a counting function argument, and integrating out t.

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- Γ cocompact: Using the bound $(*)$, the claimed bound $O_\Gamma(k)$ follows from bounding $K_k^\Gamma(t;z,w)$ for $z=w$, using a counting function argument, and integrating out t.
- Γ cofinite, but not cocompact: In this case, the fundamental domain is subdivided into small cuspidal neighborhoods and the resulting complement, which is compact. The bound $O_\Gamma(k^{3/2})$ results from z ranging in the cuspidal neighborhood and the sum over the parabolic subgroup associated to the cusp in question.

Sup-norm bounds on average in the Siegel modular case

Sup-norm bounds in the Siegel modular case

Siegel upper half-space and real symplectic group

- $\mathbb{H}_n := \{Z \in \mathrm{Sym}_n(\mathbb{C}) \mid Z = X + iY,\ Y > 0\}$,
  Siegel upper half-space of degree n.

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Siegel upper half-space and real symplectic group

- $\mathbb{H}_n := \{Z \in \mathrm{Sym}_n(\mathbb{C}) \mid Z = X + iY,\ Y > 0\}$, Siegel upper half-space of degree n.
- $\mathrm{Sp}_n(\mathbb{R}) := \{g \in \mathrm{M}_{2n}(\mathbb{R}) \mid g^t J_n g = J_n\}$ with $J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$, real symplectic group of degree n.

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- We have: $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R}) \iff$ $A^t C = C^t A, B^t D = D^t B, A^t D - C^t B = E_n$.

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  $A^t C = C^t A, B^t D = D^t B, A^t D - C^t B = E_n$.
- $\mathrm{Sp}_n(\mathbb{R})$ acts on $\mathbb{H}_n$ by fractional linear transformations:
  $Z \mapsto (AZ + B)(CZ + D)^{-1} \quad (Z \in \mathbb{H}_n)$.

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  $Z \mapsto (AZ + B)(CZ + D)^{-1} \quad (Z \in \mathbb{H}_n)$.
- $\mu_n(Z) = \frac{\bigwedge_{1 \le j \le k \le n} \mathrm{d}x_{j,k} \wedge \mathrm{d}y_{j,k}}{\det(Y)^{n+1}} \quad (Z = (z_{j,k} = x_{j,k} + iy_{j,k}))$
  is the invariant measure on $\mathbb{H}_n$.

Sup-norm bounds in the Siegel modular case

Siegel modular forms

- $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ arithmetic subgroup.

Sup-norm bounds in the Siegel modular case

Siegel modular forms

- $\Gamma \subset \mathrm{Sp}_n(\mathbb{R})$ arithmetic subgroup.
- $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ Siegel modular form of weight k and degree n for Γ, if
  - f is holomorphic,
  - $f\big((AZ+B)(CZ+D)^{-1}\big)\det(CZ+D)^{-k} = f(Z)$ for all $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$,
  - growth condition at the cusps if $n = 1$.

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- $S_{k,n}(\Gamma)$ $\mathbb{C}$-vector space of Siegel cusp forms of weight k and degree n for Γ.

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- $S_{k,n}(\Gamma)$ $\mathbb{C}$-vector space of Siegel cusp forms of weight k and degree n for Γ.
- Petersson inner product on $S_{k,n}(\Gamma)$ defined by

$$\langle f, g \rangle := \int_{\Gamma\backslash\mathbb{H}_n} f(Z)\,\overline{g(Z)}\,\det(Y)^k\,\mu_n(Z).$$

Sup-norm bounds in the Siegel modular case

Definition of the quantity $B_{k,n}^{\Gamma}(Z)$

- Let $\{f_1, \ldots, f_d\}$ be an orthonormal basis of $S_{k,n}(\Gamma)$.
- In analogy to the elliptic modular case, we put

$$B_{k,n}^{\Gamma}(Z) := \sum_{j=1}^{d} |f_j(Z)|^2 \det(Y)^k.$$

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$$B_{k,n}^{\Gamma}(Z) := \sum_{j=1}^{d} |f_j(Z)|^2 \det(Y)^k.$$

- We are interested in optimal upper bounds for $B_{k,n}^{\Gamma}(Z)$ as Z ranges through $\mathbb{H}_n$.
- By the Γ-invariance of $B_{k,n}^{\Gamma}(Z)$, we can restrict our considerations to a fundamental domain $\mathcal{F}_{\Gamma} \subset \mathbb{H}_n$ by the action of Γ.

Sup-norm bounds in the Siegel modular case

Conjectures for the supremum of the quantity $B_{k,n}^{\Gamma}(Z)$

By the results obtained in the 1-dimensional case, the following two conjectures arise:

Conjecture 1. If the subgroup Γ is cocompact and $k \geq n+1$, we have

$$\sup_{Z \in \mathcal{F}_\Gamma} B_{k,n}^{\Gamma}(Z) = O_\Gamma(k^{n(n+1)/2}).$$

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Conjecture 2. If the subgroup Γ is is cofinite, but not cocompact, and $k \geq n+1$, we have

$$\sup_{Z \in \mathcal{F}_\Gamma} B_{k,n}^{\Gamma}(Z) = O_\Gamma(k^{3n(n+1)/4}).$$

Sup-norm bounds in the Siegel modular case

We have the following results:

Result 1. If the subgroup Γ is cocompact and $k \geq n+1$, then Conjecture 1 holds for $n = 2$. The general case holds if a certain Chebyshev-type inequality $(*)$ holds globally.

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Result 1. If the subgroup Γ is cocompact and $k \geq n+1$, then Conjecture 1 holds for $n = 2$. The general case holds if a certain Chebyshev-type inequality $(*)$ holds globally.

Result 2. If the subgroup Γ is cofinite, but not cocompact, and $k \geq n+1$, then Conjecture 2 holds for $n = 2$. The general case again holds if a certain Chebyshev-type inequality $(*)$ holds globally.

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Result 2. If the subgroup Γ is cofinite, but not cocompact, and $k \geq n+1$, then Conjecture 2 holds for $n = 2$. The general case again holds if a certain Chebyshev-type inequality $(*)$ holds globally.

Remark. J. Cogdell has unconditionally shown Conjecture 1 provided that $\Gamma = \mathrm{Sp}_n(\mathbb{Z})$ and $k > 2n$. V. Blomer has provided optimal sup-norm bounds for individual Siegel modular forms that are Ikeda lifts.

Strategy of proof

Strategy of proof

Some more notation

- Let $\mathcal{V}_{k,n}(\Gamma)$ denote the space of functions $\varphi\colon \mathbb{H}_n \longrightarrow \mathbb{C}$ satisfying

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- $\Delta_k := -\mathrm{tr}\left(Y\left(\left(Y\frac{\partial}{\partial X}\right)^t \frac{\partial}{\partial X} + \left(Y\frac{\partial}{\partial Y}\right)^t \frac{\partial}{\partial Y}\right) - ikY\frac{\partial}{\partial X}\right).$

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induced by the assignment $f(Z) \mapsto f(Z)\det(Y)^{k/2}$.

Strategy of proof

Consequences

- If $\varphi \in \mathcal{H}_{k,n}(\Gamma)$ is an eigenfunction of Δ_k, hence a Maass form of weight $k/2$ and degree n for Γ, i.e., $\Delta_k \varphi(z) = \lambda\, \varphi(z)$, then we have $\lambda \geq nk(n+1-k)/4$.
- Furthermore, we find the isomorphism

$$S_{k,n}(\Gamma) \cong \ker\big(\Delta_k - nk(n+1-k)/4\big),$$

induced by the assignment $f(Z) \mapsto f(Z)\det(Y)^{k/2}$.
- This shows that $B^{\Gamma}_{k,n}(Z)$ can be seen as part of the spectral expansion of the heat kernel $K^{\Gamma}_{k,n}(t; Z, W)$ of weight k and degree n for Γ on the diagonal; here $t \in \mathbb{R}_{>0}$ and $Z, W \in \mathbb{H}$.
- More precisely, for $k \geq n+1$ and $t \in \mathbb{R}_{>0}$, we have

$$e^{-nk(n+1-k)t/4} B^{\Gamma}_{k,n}(Z) \leq K^{\Gamma}_{k,n}(t; Z, Z).$$

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Heat kernel for Γ

Concluding the proof

The heat kernel of weight k and degree n for Γ

Heat kernel of weight k and degree n for Γ

Some Lie algebraic background in this context

- Let $\mathfrak{g}_0 = \mathfrak{sp}_n(\mathbb{R})$, i.e.,

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \,\middle|\, A, B, C \in \mathrm{M}_n(\mathbb{R}) : \; B = B^t, \; C = C^t \right\}.$$

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- $\theta \colon \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$ Cartan involution induced by

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mapsto -\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}^t.$$

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- $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ resulting Cartan decomposition, where

$\mathfrak{k}_0 = (+1)$-eigenspace of θ, $\mathfrak{p}_0 = (-1)$-eigenspace of θ.

Heat kernel of weight k and degree n for Γ

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  $\mathfrak{k}_0 = (+1)$-eigenspace of θ, $\mathfrak{p}_0 = (-1)$-eigenspace of θ.

- $\mathfrak{a}_0$ maximal abelian subalgebra of $\mathfrak{p}_0$ given by

$$\mathfrak{a}_0 = \left\{ \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \,\middle|\, R = \mathrm{diag}(r_1, \ldots, r_n) : \; r_1, \ldots, r_n \in \mathbb{R} \right\}.$$

Heat kernel of weight k and degree n for Γ

Some Lie group-theoretic background in this context

- We put $G_0 := \exp(\mathfrak{g}_0)$, i.e., $G_0 = \mathrm{Sp}_n(\mathbb{R})$.

Heat kernel of weight k and degree n for Γ

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$$K_0 = \left\{ \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix} \right) \in G_0 \,\middle|\, A + iB \in U_n(\mathbb{C}) \right\} = G_0 \cap O_{2n}(\mathbb{R}).$$

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- We put $A_0 := \exp(\mathfrak{a}_0)$, i.e.,

$$A_0 = \left\{ \begin{pmatrix} e^R & 0 \\ 0 & e^{-R} \end{pmatrix} \in G_0 \,\middle|\, R = \mathrm{diag}(r_1, \ldots, r_n) : \; r_1, \ldots, r_n \in \mathbb{R} \right\}.$$

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- We obtain the decomposition $G_0 = K_0 A_0 K_0$, for which the A_0-component is uniquely determined up to conjugation by the Weyl group $W(G_0, A_0) := N_{G_0}(A_0)/Z_{G_0}(A_0)$.

Heat kernel of weight k and degree n for Γ

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- We note that $G_0/K_0 \cong \mathbb{H}_n$, induced by $gK_0 \mapsto g(iE_n)$.

Heat kernel of weight k and degree n for Γ

The heat kernel of weight 0 and degree n for Γ

- For $t \in \mathbb{R}_{>0}$ and $Z, W \in \mathbb{H}_n$, the heat kernel of weight 0 and degree n for Γ is given by

$$K_{0,n}^{\Gamma}(t; Z, W) = \sum_{\gamma \in \Gamma} K_{0,n}(t; Z, \gamma W),$$

where $K_{0,n}(t; Z, W)$ is the heat kernel of weight 0 on $\mathbb{H}_n$.

Heat kernel of weight k and degree n for Γ

The heat kernel of weight 0 and degree n for Γ

- For $t \in \mathbb{R}_{>0}$ and $Z, W \in \mathbb{H}_n$, the heat kernel of weight 0 and degree n for Γ is given by

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  where $K_{0,n}(t; Z, W)$ is the heat kernel of weight 0 on $\mathbb{H}_n$.

- In order to construct $K_{0,n}(t; Z, W)$, we note that by the double transitive action of G_0, there is $g \in G_0$ such that

$$Z = g(iE_n) \quad \text{and} \quad W = g(ie^{2R}),$$

  where $R = \operatorname{diag}(r_1, \ldots, r_n)$ with $r_1, \ldots, r_n \in \mathbb{R}$.

- One is then reduced to construct $K_{0,n}(t; iE_n, ie^{2R})$.

Heat kernel of weight k and degree n for Γ

The heat kernel of weight 0 on $\mathbb{H}_n$, part I

- Via the isomorphism $G_0/K_0 \cong \mathbb{H}_n$, we recall the correspondences $K_0 \leftrightarrow iE_n$ and $e^r K_0 \leftrightarrow ie^{2R}$, where $r = \left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\right) \in \mathfrak{g}_0$ and $e^r \in G_0$.
- The heat kernel $K_{0,n}(t; iE_n, ie^{2R})$ is now given by the formula

$$K_{0,n}(t; iE_n, ie^{2R}) = \int_{\mathfrak{a}_0^\vee} e^{\lambda_\omega t}\, \phi_\lambda(e^r K_0)\, |c(\lambda)|^{-2}\, \mathrm{d}\lambda,$$

where

  - $\phi_\lambda(\cdot)$ is a so-called spherical function on G_0/K_0,
  - $\Delta_0 \phi_\lambda = \lambda_\omega \phi_\lambda$,
  - $c(\lambda)$ is Harish-Chandra's c-function for G_0.

Heat kernel of weight k and degree n for Γ

Expliciting the eigenvalue λ_ω

- Recall that

$$\mathfrak{a}_0 = \left\{ \begin{pmatrix} R & 0 \\ 0 & -R \end{pmatrix} \,\middle|\, R = \operatorname{diag}(r_1, \ldots, r_n) : \; r_1, \ldots, r_n \in \mathbb{R} \right\}.$$

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- Then, $\mathfrak{a}_0^\vee$ has basis $\{e_1, \ldots, e_n\}$ characterized for $r \in \mathfrak{a}$ by

$$e_j\left(\left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix} \right) \right) = r_j \quad (j = 1, \ldots, n).$$

Sup-norm bounds for Siegel modular forms

Jürg Kramer

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Expliciting the eigenvalue λ_ω

- Recall that

$$\mathfrak{a}_0 = \left\{ \left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix} \right) \mid R = \mathrm{diag}(r_1, \ldots, r_n) : \ r_1, \ldots, r_n \in \mathbb{R} \right\}.$$

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$$e_j\left(\left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix} \right) \right) = r_j \quad (j = 1, \ldots, n).$$

- Writing $\lambda \in \mathfrak{a}_0^\vee$ as $\lambda = \lambda_1 e_1 + \ldots + \lambda_n e_n$, we have

$$\lambda_\omega = -\frac{\sum_{j=1}^n j^2 + \sum_{j=1}^n \lambda_j^2}{4}.$$

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Expliciting Harish-Chandra's c-function

Bhanu Murti has obtained the following formula for $c(\lambda)$

$$|c(\lambda)|^{-2} = \pi^{n^2/2} \prod_{1 \le j \le n} \frac{\lambda_j}{2} \tanh\left(\frac{\lambda_j}{2}\pi\right) \times \prod_{1 \le j < k \le n} \frac{\lambda_j + \lambda_k}{2} \tanh\left(\frac{\lambda_j + \lambda_k}{2}\pi\right) \times \prod_{1 \le j < k \le n} \frac{\lambda_j - \lambda_k}{2} \tanh\left(\frac{\lambda_j - \lambda_k}{2}\pi\right).$$

Heat kernel of weight k and degree n for Γ

Expliciting $\phi_\lambda(\cdot)$ via the Flensted-Jensen formalism

- Recall that $G_0 = \mathrm{Sp}_n(\mathbb{R})$ and $K_0 = \mathrm{Sp}_n(\mathbb{R}) \cap \mathrm{O}_{2n}(\mathbb{R})$.

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- We then put
  - $G := \mathrm{Sp}_n(\mathbb{C})$,
  - $K := \mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{O}_{2n}(\mathbb{C})$,
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Heat kernel of weight k and degree n for Γ

Expliciting $\phi_\lambda(\cdot)$ via the Flensted-Jensen formalism

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  - $U := \mathrm{Sp}_n(\mathbb{C}) \cap \mathrm{U}_{2n}(\mathbb{C})$.
- The spherical function $\phi_\lambda(\cdot)$ on G_0/K_0 is then related to the spherical function $\Phi_{2\lambda}(\cdot)$ on G/U by the formula

$$\phi_\lambda(gg^tK_0)\,|c(\lambda)|^{-2} = |\pi_0(\lambda)|^2 \int_K \Phi_{2\lambda}(qgU)\,\mathrm{d}\mu(q),$$

where $\Phi_{2\lambda}(\cdot)$ is somewhat explicitly accessible, and $|\pi_0(\lambda)|$ can be computed in a similar way as $|c(\lambda)|$.

Sup-norm bounds for Siegel modular forms

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The heat kernel of weight 0 on $\mathbb{H}_n$, part II

Putting everything together and carrying out the λ-integral over $\mathfrak{a}_0^\vee$, we arrive at the formula (c_n constant depending on n)

$$K_{0,n}(t; iE_n, ie^{2R}) = c_n \frac{e^{-\sum_{j=1}^n j^2 t/4}}{t^{n^2+n/2}} \int_K \frac{\varepsilon(\varrho(r,q)) e^{-\sum_{j=1}^n \varrho_j(r,q)^2/t}}{\delta(\varrho(r,q))} \mathrm{d}\mu(q)$$

where

Heat kernel of weight k and degree n for Γ

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where

- $\varrho(r,q) = \begin{pmatrix} P(r,q) & 0 \\ 0 & -P(r,q) \end{pmatrix}$ with $P(r,q) = \mathrm{diag}(\varrho_j(r,q))$,
- subject to $qe^r\overline{q}^t = ue^\varrho\overline{u}^t$ with $q \in K$ and $u \in U$,

Heat kernel of weight k and degree n for Γ

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- subject to $qe^r\overline{q}^t = ue^\varrho \overline{u}^t$ with $q \in K$ and $u \in U$,
- $\varepsilon(\varrho) = \prod_{1 \le j \le n} \varrho_j \prod_{1 \le j < k \le n} (\varrho_j + \varrho_k) \prod_{1 \le j < k \le n} (\varrho_j - \varrho_k)$,
- $\delta(\varrho) = \prod_{1 \le j \le n} \sinh(\varrho_j) \prod_{1 \le j < k \le n} \sinh(\varrho_j \pm \varrho_k)$.

Heat kernel of weight k and degree n for Γ

The heat kernel of weight k on $\mathbb{H}_n$

The heat kernel $K_{k,n}(t; iE_n, ie^{2R})$ of weight k on $\mathbb{H}_n$ is immediately obtained from the previous formula by inserting the factor $\det(h(q))^{2k}$, i.e.,

$$K_{k,n}(t; iE_n, ie^{2R}) =$$

$$c_n \frac{e^{-\sum_{j=1}^n j^2 t/4}}{t^{n^2+n/2}} \int_K \frac{\varepsilon(\varrho(r,q)) e^{-\sum_{j=1}^n \varrho_j(r,q)^2/t}}{\delta(\varrho(r,q))} \det(h(q))^{2k} \mathrm{d}\mu(q),$$

where the matrix $h(q) \in \mathrm{M}_n(\mathbb{C})$ is obtained as follows:

- Write $q \in K$ as $q = q_0\, q_h$ with $q_0 \in K_0$ and

  $$q_h = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \text{ is hermitian orthogonal.}$$

- Then, we obatin $h(q) = A + iB$, which is hermitian.

Concluding the proof

Concluding the proof

Assembling the crucial pieces

- Recall that for $k \geq n+1$, we have the bound

$$e^{-nk(n+1-k)t/4} B_{k,n}^{\Gamma}(Z) \leq K_{k,n}^{\Gamma}(t; Z, Z).$$

Concluding the proof

Assembling the crucial pieces

- Recall that for $k \geq n+1$, we have the bound

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- Here, the heat kernel of weight k and degree n for Γ is given by

$$K_{k,n}^{\Gamma}(t; Z, Z) = \sum_{\gamma \in \Gamma} K_{k,n}(t; Z, \gamma Z).$$

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$$K_{k,n}^{\Gamma}(t; Z, Z) = \sum_{\gamma \in \Gamma} K_{k,n}(t; Z, \gamma Z).$$

- Choosing $g \in G_0$ such that $Z = g(iE_n)$ and $\gamma Z = g(ie^{2R})$, we have from the previous slide

$$K_{k,n}(t; Z, \gamma Z) = K_{k,n}(t; iE_n, ie^{2R}) =$$

$$c_n \frac{e^{-\sum_{j=1}^n j^2 t/4}}{t^{n^2+n/2}} \int_K \frac{\varepsilon(\varrho(r,q)) e^{-\sum_{j=1}^n \varrho_j(r,q)^2/t}}{\delta(\varrho(r,q))} \det(h(q))^{2k} \mathrm{d}\mu(q).$$

Concluding the proof

The crucial inequality With the above notations, we expect that the function $\det(h(q))^{2k}$ satisfies the following inequality

$$\det(h(q))^{2k} \leq \frac{\exp\left(k\sum_{j=1}^{n}|\varrho_j|\right)}{\prod_{j=1}^{n}\cosh^k(r_j)}, \qquad (**)$$

where

- $r = \left(\begin{smallmatrix} R & 0 \\ 0 & -R \end{smallmatrix}\right)$ with $R = \mathrm{diag}(r_1, \ldots, r_n)$,
- $\varrho(r,q) = \left(\begin{smallmatrix} P(r,q) & 0 \\ 0 & -P(r,q) \end{smallmatrix}\right)$ with $P(r,q) = \mathrm{diag}(\varrho_j(r,q))$,
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- subject to $qe^r\overline{q}^t = ue^{\varrho}\overline{u}^t$ with $q \in K$ and $u \in U$.

Proposition. The inequality $(**)$ holds for $n = 1$ and $n = 2$.

Remark. We expect that the inequality $(**)$ holds for any $n \in \mathbb{N}_{>0}$, since we can show that the right-hand side is always at least a local maximum.

Concluding the proof

The last step Assuming the inequality $(**)$, we are left to estimate the integral

$$c_n \frac{e^{-\sum_{j=1}^n j^2 t/4}}{t^{n^2+n/2}} \sum_{\gamma\in\Gamma} \int_K \frac{\varepsilon(\varrho) e^{-\sum_{j=1}^n \varrho_j(r,q)^2/t + k\sum_{j=1}^n |\varrho_j|}}{\delta(\varrho(r,q)) \prod_{j=1}^n \cosh^k(r_j)} \mathrm{d}\mu(q).$$

Changing the variables from (r,q) to (ϱ,u), then using a counting function argument (to take care of the sum over $\gamma\in\Gamma$), and integrating out u, we complete the proof by distinguishing the cases when Γ is cocompact and cofinite, respectively:

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Changing the variables from (r,q) to (ϱ,u), then using a counting function argument (to take care of the sum over $\gamma\in\Gamma$), and integrating out u, we complete the proof by distinguishing the cases when Γ is cocompact and cofinite, respectively:

- If Γ is cocompact, the calculation of the remaining ρ-integral leads to the bound $O_\Gamma(k^{n(n+1)/2})$.

Concluding the proof

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$$c_n \frac{e^{-\sum_{j=1}^n j^2 t/4}}{t^{n^2+n/2}} \sum_{\gamma\in\Gamma} \int_K \frac{\varepsilon(\varrho) e^{-\sum_{j=1}^n \varrho_j(r,q)^2/t + k\sum_{j=1}^n |\varrho_j|}}{\delta(\varrho(r,q)) \prod_{j=1}^n \cosh^k(r_j)} \mathrm{d}\mu(q).$$

Changing the variables from (r,q) to (ϱ,u), then using a counting function argument (to take care of the sum over $\gamma\in\Gamma$), and integrating out u, we complete the proof by distinguishing the cases when Γ is cocompact and cofinite, respectively:

- If Γ is cocompact, the calculation of the remaining ρ-integral leads to the bound $O_\Gamma(k^{n(n+1)/2})$.
- If Γ is cofinite, but not cocomapct, the argument has to be modified in the cuspidal neighborhoods, which leads to the bound $O_\Gamma(k^{3n(n+1)/4})$.

Concluding the proof

The two main theorems. From the above analysis, we arrive at the following results:

Theorem 1. If $n \in \{1, 2\}$, the subgroup Γ is cocompact, and $k \geq n + 1$, we have

$$\sup_{Z \in \mathcal{F}_\Gamma} B_{k,n}^\Gamma(Z) = O_\Gamma(k^{n(n+1)/2}).$$

Concluding the proof

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Theorem 2. If $n \in \{1, 2\}$, the subgroup Γ is cofinite, but not cocompact, and $k \geq n + 1$, we have

$$\sup_{Z \in \mathcal{F}_\Gamma} B_{k,n}^\Gamma(Z) = O_\Gamma(k^{3n(n+1)/4}).$$

Congratulation

Dear Steve,

I wish you all the best for your 70th birthday:

good health,

continuing mathematical productivity,

enjoying life!