False theta functions and their modularity properties

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Outline

1. Modular forms and mock modular forms

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2. False theta functions

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Modularity

Definition:

$f : \mathbb{H} \to \mathbb{C}$ holomorphic is modular of weight k if for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$

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plus growth condition

Fourier expansion ($q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$)

$$f(\tau) = \sum_{n\in\mathbb{Z}} c(n)q^n$$

Examples

- Dedekind η-function:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n)$$

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- Theta function:

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$$\Theta(\tau)^3 =: \sum_{n\geq 0} r(n)q^n.$$

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Then

$$r(n) = 12\big(H(4n) - 2H(n)\big)$$

with

$$H(n) := \#\{\text{equivalence classes of integral binary quadratic forms of discriminant } n\}.$$

Harmonic Maass forms

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$F : \mathbb{H} \to \mathbb{C}$ real-analytic is a weight k harmonic Maass form if it is modular of weight k and

J. Bruinier

J. Funke

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$$\Delta_k(F)=0$$

with $(\tau=\tau_1+i\tau_2)$

$$\Delta_k:=-\tau_2^2\left(\frac{\partial^2}{\partial\tau_1^2}+\frac{\partial^2}{\partial\tau_2^2}\right)+ik\tau_2\left(\frac{\partial}{\partial\tau_1}+i\frac{\partial}{\partial\tau_2}\right)$$

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$$\widehat{E}_2(\tau) := \underset{\substack{\uparrow \\ \text{quasimodular}}}{E_2}(\tau) - \frac{3}{\pi\tau_2}$$

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- Class number generating function:

$$\widehat{\mathcal{H}}(\tau) := \sum_{\substack{n\geq0 \\ n\equiv0,3 \pmod 4}} H(n)q^n + \frac{i}{8\sqrt{2}\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\Theta(w)}{(-i(\tau+w))^{\frac{3}{2}}} dw.$$

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Natural splitting

$\mathcal{F}$ harmonic Maass form

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with

$$\mathcal{F}^+(\tau) := \sum_{n \gg -\infty} c^+(n) q^n$$

$$\mathcal{F}^-(\tau) := \sum_{n \geq 1} c^-(n) \Gamma(k-1; 4\pi |n| \tau_2) q^n$$

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Alternative representation

The non-holomorphic part has the shape

$$\int_{-\bar{\tau}}^{i\infty} g(w)(\tau+w)^{2-k}dw.$$

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$$\int_{-\bar{\tau}}^{i\infty} \underset{\substack{\uparrow \\ \text{modular, weight } 2-k}}{g(w)}(\tau+w)^{2-k}dw.$$

Ramanujan's last letter

> "I am extremely sorry for not writing you a single letter up to now. I recently discovered very interesting functions which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions they enter into mathematics as beautifully as the theta functions. I am sending you with this letter some examples."

S. Ramanujan

Mock theta functions

These mock theta functions are 22 peculiar q-series.

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Example:

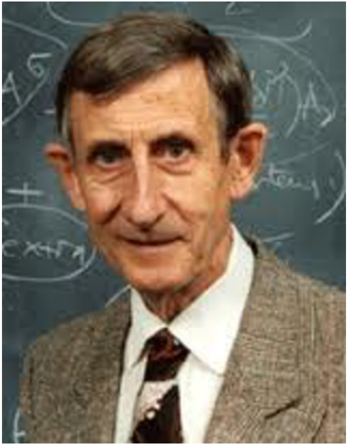
$$f(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(-q;q)_n^2}$$

with

$$(a;q)_n := \prod_{m=0}^{n-1} (1 - aq^m)$$

Dyson's challenge for the future

> "The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta functions of Jacobi. This remains a challenge for the future..."



F. Dyson

Mock modularity of $f(q)$

Theorem (Zwegers)

The function $f(q)$ is a mock modular form.

S. Zwegers

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Rogers false theta functions

Wrong sgn-factors prevent modularity.

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Example:

$$\sum_{n\in\mathbb{Z}}(-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\left(n+\frac{1}{2}\right)^2}$$


L. J. Rogers

Quantum modular forms

Definition:

$f : \mathcal{Q} \to \mathbb{H}$ ($\mathcal{Q} \subset \mathbb{Q}$) is a quantum modular form of weight k if for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\tau) - (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

is “nice”.

D. Zagier

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$$f : \overset{\text{quantum set}}{\overset{\downarrow}{\mathcal{Q}}} \to \mathbb{H}$$

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Examples: false theta functions

D. Zagier

Strange function

Let

$$\widetilde{\eta}(\tau) := -\frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right)|n|q^{\frac{n^2-1}{24}}$$

M. Kontsevich

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“Strange identity”:

$$\widetilde{\eta}(\tau)\text{“}=\text{”}K(q) := 1 + \sum_{n\geq 1}(q;q)_n$$

M. Kontsevich

Vertex algebras

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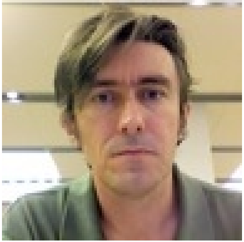
- Characters of rational vertex algebra are modular functions.
- Singlet vertex algebras are non-rational, characters are 1-dimensional false theta functions.
- Triplet algebras lead to higher-dimensional theta functions.

Vertex algebras

Class of functions: $(0 \leq s < N \in \mathbb{N})$

$$\frac{F_{N-s,N}(N\tau)}{\eta(\tau)}$$



A. Milas

Vertex algebras

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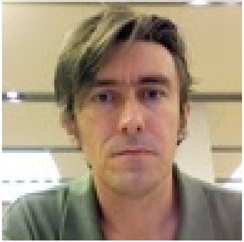
$$\frac{F_{N-s,N}(N\tau)}{\eta(\tau)}$$

with

$$F_{j,N}(\tau) :\doteq \sum_{n \equiv j \pmod{2N}} \operatorname{sgn}(n) q^{\frac{n^2}{4N}}$$



A. Milas

False theta functions in combinatorics

Definition:

Sequence $\{a_j\}_{j=1}^{s}$ with

$$a_1 \leq a_2 \leq \ldots \leq a_k \geq a_k \geq \ldots \geq a_s$$

and $a_1 + \ldots + a_s = n$ is a unimodal sequence.

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Generating function:

$$U(q) := \sum_{n\geq 0} u(n)q^n = \frac{1}{(q;q)_\infty^2} \sum_{n\geq 1} (-1)^{n+1} q^{\frac{n(n+1)}{2}}$$

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Two-variable false theta function

Define

$$\psi(z;\tau) := i \sum_{n\in\mathbb{Z}} (-1)^n \operatorname{sgn}\left(n+\frac{1}{2}\right) q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \zeta^{n+\frac{1}{2}}.$$

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Remark:

Removing the sgn yields a Jacobi form. The sgn breaks the inversion property.

Jacobi forms

Definition:

Holomorphic Jacobi forms of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}_0$ are holomorphic functions

M. Eichler

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$\phi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ satisfying, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\lambda, \mu \in \mathbb{Z}$,

$$\phi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k e^{\frac{2\pi icmz^2}{c\tau+d}} \phi(z;\tau),$$

$$\phi(z+\lambda\tau+\mu;\tau) = e^{-2\pi im\left(\lambda^2\tau+2\lambda z\right)}\phi(z;\tau).$$

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plus growth conditions

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Idea:

“Complete” ψ to obtain a function transforming like Jacobi forms.

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"Complete" ψ to obtain a function transforming like Jacobi forms.

Completion: $w \in \mathbb{H}$, $z_2 := \mathrm{Im}(z)$

$$\widehat{\psi}(z;\tau,w) := i \sum_{n\in\mathbb{Z}} \mathrm{erf}\left(-i\sqrt{\pi i(w-\tau)}\left(n+\frac{1}{2}+\frac{z_2}{\tau_2}\right)\right) \times (-1)^n q^{\frac{1}{2}\left(n+\frac{1}{2}\right)^2} \zeta^{n+\frac{1}{2}}$$

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Note that, for $-\frac{1}{2} < \frac{z_2}{\tau_2} < \frac{1}{2}$ and $\varepsilon > 0$,

$$\lim_{t\to\infty} \widehat{\psi}(z;\tau,\tau+it+\varepsilon) = \psi(z;\tau)$$

Modularity

Theorem 1 (B.-Nazaroglu)

The function $\widehat{\psi}$ transforms like a Jacobi form.

C. Nazaroglu

Modularity

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Sketch of proof:

Poisson summation.

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Partitions

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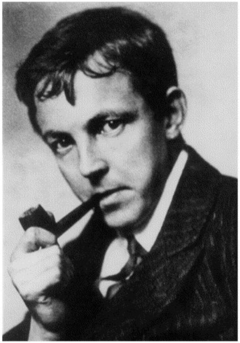
L. Euler

Growth of $p(n)$

$$p(10) = 42$$



G. Hardy

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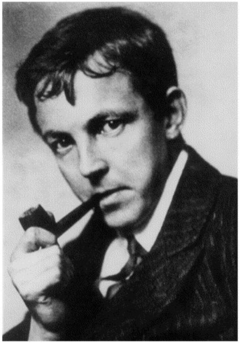
$$p(50) = 204226$$



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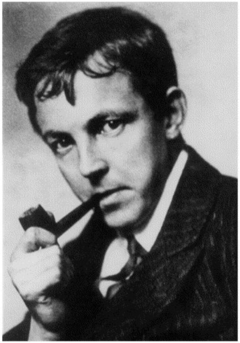

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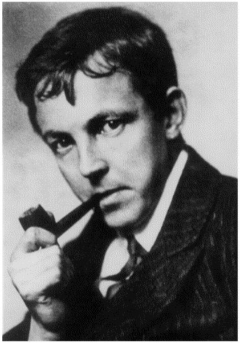

G. Hardy

Asymptotic behavior: (Hardy–Ramanujan)

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}} \qquad (n \to \infty)$$

Exact formula

Kloosterman sums:

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Bessel function of order α:

$$I_\alpha(x) := \sum_{m\geq 0} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

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$$I_\alpha(x) := \sum_{m\geq 0} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}$$

Rademacher formula:

$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k\geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

Unimodal sequences

Asymptotics: (Auluck, Wright)

$$u(n) \sim \frac{1}{8 \cdot 3^{\frac{3}{4}} n^{\frac{5}{4}}} e^{2\pi\sqrt{\frac{n}{3}}} \qquad (n \to \infty)$$

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Notation: $K_k(n)$, $K_k(n, r)$ Kloosterman sums

Unimodal sequences

Theorem 2 (B.-Nazaroglu)

We have

$$u(n)=\frac{2\pi}{12n-1}\sum_{k\geq 1}\frac{K_k(n)}{k}I_2\left(\frac{\pi}{3k}\sqrt{12n-1}\right)$$

$$-\frac{\pi}{2^{\frac{3}{4}}\sqrt{3}(24n+1)^{\frac{3}{4}}}\sum_{k\geq 1}\sum_{r \pmod{2k}}\frac{K_k(n,r)}{k^2}$$

$$\times\int_{-1}^{1}\left(1-x^2\right)^{\frac{3}{4}}\cot\left(\frac{\pi}{2k}\left(\frac{x}{\sqrt{6}}-r-\frac{1}{2}\right)\right)I_{\frac{3}{2}}\left(\frac{\pi}{3\sqrt{2}k}\sqrt{\left(1-x^2\right)\left(24n+1\right)}\right)dx.$$

Unimodal sequences

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Idea of proof: Use Circle Method and modular completion.

Quantum modular forms

Recall

$$F_{j,N}(\tau) \doteq \sum_{n \equiv j \pmod{2N}} \operatorname{sgn}(n) q^{\frac{n^2}{4N}}.$$

Quantum modular forms

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Asymptotics

$$F_{j,N}\left(it+\frac{h}{k}\right) \sim \sum_{m\geq 0} a_{h,k}(m)t^m \quad \left(t\to 0^+\right),$$

$$F^*_{j,N}\left(it-\frac{h}{k}\right) \sim \sum_{m\geq 0} a_{h,k}(m)(-t)^m \quad \left(t\to 0^+\right),$$

Quantum modular forms

where

$$F^*_{j,N}(\tau) :\doteq \sum_{\substack{n\in\mathbb{Z} \\ n\equiv j \pmod{2N}}} \operatorname{sgn}(n)\Gamma\left(\frac{1}{2};\frac{\pi n^2\tau_2}{p}\right) q^{-\frac{n^2}{4N}}.$$

Quantum modular forms

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Note that

$$F_{j,N}^*(\tau) \doteq \int_{-\overline{\tau}}^{i\infty} \frac{f_{j,N}(w)}{(-i(w+\tau))^{\frac{1}{2}}} dw,$$

where

$$f_{j,N}(\tau) := \frac{1}{2N} \sum_{n\equiv j \pmod{2N}} nq^{\frac{n^2}{4N}}.$$

False transformation properties

Theorem 3 (B.-Nazaroglu)

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$F_{j,N}(\tau) - \operatorname{sgn}(c\tau_1 + d)(c\tau + d)^{-\frac{1}{2}} \sum_{r=1}^{N-1} \psi_{j,r}\left(M^{-1}\right) F_{r,N}\left(\frac{a\tau + b}{c\tau + d}\right) = -i\sqrt{2N} \int_{-\frac{d}{c}}^{i\infty} \frac{f_{j,N}(\mathfrak{z})}{\sqrt{-i(\mathfrak{z} - \tau)}} d\mathfrak{z}.$$

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Corollary 4

$F_{j,N}$ are vector-valued quantum modular forms of weight $\frac{1}{2}$ with quantum set $\mathbb{Q}$.

Outline

1. Modular forms and mock modular forms
2. False theta functions
3. Modularity properties of false theta functions
4. Applications
5. Higher-dimensional false theta functions

Higher-dimensional theta functions

- Arise from triplet algebras

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- A typical example:

$$F(q) := \sum_{\boldsymbol{\alpha} \in \mathscr{S}} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n} \in \mathbb{N}_0^2 + \boldsymbol{\alpha}} q^{Q(\boldsymbol{n})} + \frac{1}{2} \sum_{n \in \mathbb{Z}} \operatorname{sgn}\left(n + \frac{1}{N}\right) q^{\left(n + \frac{1}{N}\right)^2},$$

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where $\boldsymbol{n} = (n_1, n_2)$, $\quad Q(\boldsymbol{n}) := 3n_1^2 + 3n_1n_2 + n_2^2$,

$$\mathscr{S} := \left\{ \left(1-\tfrac{1}{N}, \tfrac{2}{N}\right), \left(\tfrac{1}{N}, 1-\tfrac{2}{N}\right), \left(1, \tfrac{1}{N}\right), \left(0, 1-\tfrac{1}{N}\right), \right.$$
$$\left. \left(\tfrac{1}{N}, 1-\tfrac{1}{N}\right), \left(1-\tfrac{1}{N}, \tfrac{1}{N}\right) \right\},$$

Higher-dimensional theta functions

$$\varepsilon(\boldsymbol{\alpha}) := \begin{cases} -2 & \text{if } \boldsymbol{\alpha} \in \left\{\left(1-\frac{1}{N}, \frac{2}{N}\right), \left(\frac{1}{N}, 1-\frac{2}{N}\right)\right\}, \\ 1 & \text{otherwise.} \end{cases}$$

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Note that

$$F(q) = \frac{1}{2} \sum_{\boldsymbol{\alpha} \in \mathcal{S}^*} \varepsilon(\boldsymbol{\alpha}) \sum_{\boldsymbol{n} \in \mathbb{Z}^2 + \boldsymbol{\alpha}} \operatorname{sgn}(n_2) \left(\operatorname{sgn}(n_1) + \operatorname{sgn}(n_2)\right) q^{Q(\boldsymbol{n})},$$

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Jacobi forms

Idea: View F as coefficient of a (meromorphic) Jacobi form.

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Classical fact: Fourier coefficients of holomorphic Jacobi forms are modular forms.

Jacobi forms

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Classical fact: Fourier coefficients of holomorphic Jacobi forms are modular forms.

Fourier coefficients: $(z_0 \in \mathbb{C})$

$$h_\ell(\tau) = h_{\ell,z_0}(\tau) := q^{-\frac{\ell^2}{4m}} \int_{z_0}^{z_0+1} \phi(z;\tau) e^{-2\pi i \ell z} dz$$

Fourier coefficients of Jacobi forms

Theorem 5 (B.-Rolen-Zwegers)

Let $m \in -\frac{1}{2}\mathbb{N}$, $\phi: \ \mathbb{C} \to \mathbb{C}$ meromorphic function satisfying for $\lambda, \mu \in \mathbb{Z}$

$$\phi(z + \lambda\tau + \mu) = (-1)^{2m\mu+\lambda\varepsilon} e^{-2\pi i m\left(\lambda^2\tau+2\lambda z\right)}\phi(z)$$

with $\varepsilon \in \{0, 1\}$.

L. Rolen

Fourier coefficients of Jacobi forms

We have for $\ell \in \mathbb{Z} + m$

$$h_{\ell,z_0}(\tau) = \sum_{w \in S_{z_0,\tau}} \sum_{n \in \mathbb{N}} \frac{D_{-n,w}(\tau)}{(n-1)!} \left[\left(\frac{1}{2\pi i} \frac{\partial}{\partial z} \right)^{n-1} \vartheta^{+}_{\ell,\varepsilon,-m}(z;\tau) \right]_{z=w},$$

Fourier coefficients of Jacobi forms

We have for $\ell \in \mathbb{Z} + m$

$$h_{\ell,z_0}(\tau) = \sum_{w \in \underset{\substack{\uparrow\\ \text{set of representative of poles of } \phi}}{S_{z_0,\tau}}} \sum_{n \in \mathbb{N}} \frac{D_{-n,w}(\tau)}{(n-1)!} \left[\left(\frac{1}{2\pi i} \frac{\partial}{\partial z} \right)^{n-1} \vartheta^{+}_{\ell,\varepsilon,-m}(z;\tau) \right]_{z=w},$$

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with the partial theta function

$$\vartheta^+_{\ell,\varepsilon,M}(z;\tau) := \sum_{n \geq 0} (-1)^{n\varepsilon} q^{\frac{2Mn-\ell^2}{4M}} \zeta^{2Mn-\ell}.$$

F and Jacobi forms

Theorem 6 (B.-Kaszian-Milas-Zwegers)

We have

$$F(q)=\frac{\eta(\tau)^5}{\eta(2\tau)}\underset{\substack{\uparrow\\ \text{constant term}}}{\mathrm{CT}}_{[\zeta_1,\,\zeta_2]}\frac{\vartheta(z_1;2\tau)\vartheta(z_2;2\tau)\vartheta(z_1+z_2;2\tau)}{\vartheta(z_1;\tau)\vartheta(z_2;\tau)\vartheta(z_1+z_2;\tau)},$$


J. Kaszian

F and Jacobi forms

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where $\zeta_j := e^{2\pi i z_j}$ satisfy $|q| < |\zeta_j| < 1$, $|q| < |\zeta_1\zeta_2| < 1$ and

$$\vartheta(z;\tau) := -iq^{\frac{1}{8}}\zeta^{-\frac{1}{2}} \prod_{n\geq 1} (1-q^n)\left(1-\zeta q^{n-1}\right)\left(1-\zeta^{-1}q^n\right).$$

More general characters

Character of vacuum vertex algebra: $\mathfrak{g}$ simple Lie algebra

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$$\overset{\substack{\text{universal vertex operator algebra}\\ \downarrow}}{\mathrm{ch}[V_k(\mathfrak{g})](\zeta;q)} = \frac{1}{(q;q)_\infty^n \prod_{\alpha\in\Delta_+}(\zeta^\alpha q;q)_\infty \prod_{\alpha\in\Delta_+}(\zeta^{-\alpha} q;q)_\infty}.$$

More general characters

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charge variable

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Character of parafermionic vertex algebra:

$$(q;q)_\infty^n \mathrm{CT}_{[\zeta]}\mathrm{ch}[V_k(\mathfrak{g})](\zeta;q)$$

A simple example

Goal: Characterize CT for Lie algebras of type A_2, B_2.

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Example: $\mathfrak{g} = \mathrm{sl}_2$:

$$\mathrm{CT}_{[\zeta]}\left(\frac{1}{(\zeta q;q)_\infty(\zeta^{-1}q;q)_\infty}\right) = \frac{1}{(q;q)_\infty^2}\left(-1+2\sum_{n\geq 0}(-1)^n q^{\frac{n(n+1)}{2}}\right)$$

A_2 parafermionic character

Let

$$G(\zeta) := \frac{1}{\left(\zeta_1 q, \zeta_1^{-1} q, \zeta_2 q, \zeta_2^{-1} q, \zeta_1\zeta_2 q, \zeta_1^{-1}\zeta_2^{-1} q; q\right)_\infty}.$$

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Relation to Jacobi forms:

$$G(\zeta) = iq^{\frac{1}{4}}\eta^3 \frac{\zeta_1^{-1}\zeta_2^{-1}(1-\zeta_1)(1-\zeta_2)(1-\zeta_1\zeta_2)}{\vartheta(z_1)\vartheta(z_2)\vartheta(z_1+z_2)}$$

Constant terms:

Let

$$G_0(\tau) := 3\sum_{n\in\mathbb{Z}} |n| q^{n^2} - 6q^{-\frac{1}{4}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} |n| q^{n^2},$$

$$\Psi(\tau) := \sum_{\boldsymbol{n}\in\mathbb{Z}^2+\left(\frac{1}{3},\frac{1}{3}\right)} \operatorname{sgn}(n_1)\operatorname{sgn}(n_2) n_1 q^{Q_A(\boldsymbol{n})}, \text{ where } Q_A(\boldsymbol{n}) := n_1^2 + n_1 n_2 + n_2^2.$$

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Theorem 7 (B.-Kaszian-Milas-Nazaroglu)

For $|q| < |\zeta_1|, |\zeta_2|, |\zeta_1\zeta_2| < 1$ *we have*

$$\mathrm{CT}_{[\zeta]} G(\zeta) = \frac{q^{\frac{1}{4}}}{\eta(\tau)^6} + \frac{q^{\frac{1}{4}}}{\eta(\tau)^6} G_0(\tau) + \frac{9q^{-\frac{1}{12}}}{\eta(\tau)^6}\Psi(\tau).$$

Regularized integrals

Let

$$h(\boldsymbol{w}) := \vartheta_{3,1}^{[1]}(w_1)\vartheta_{1,1}(w_2) - \vartheta_{3,2}^{[1]}(w_1)\vartheta_{1,0}(w_2),$$

Regularized integrals

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where for $m \in \mathbb{Z}$

$$\vartheta_{m,r}^{[k]}(\tau) := \left[\left(\frac{1}{4\pi i m}\frac{\partial}{\partial z}\right)^k \vartheta_{m,r}(z;\tau)\right]_{z=0}, \text{ with } \vartheta_{m,r}(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{r}{2m}} q^{mn^2}\zeta^{2mn}.$$

Regularized integrals

Regularized integral: $(w_1 \in \mathbb{H} \setminus \{\tau\})$

$$ {}_{*}\!\int_{\tau}^{w_1} \frac{f(w_2)}{(i(w_2-\tau))^{\frac{3}{2}}} dw_2 := \lim_{\mathfrak{z}\to\tau} \left(\int_{\mathfrak{z}}^{w_1} \frac{f(w_2)}{(i(w_2-\tau))^{\frac{3}{2}}} dw_2 + 2i \frac{f(\tau)}{\sqrt{i(\mathfrak{z}-\tau)}} \right) $$

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Integral representation:

Lemma 8

We have

$$ \Psi(\tau) = \frac{\sqrt{3}}{2\pi} \int_{\tau}^{\tau+i\infty} {}_{*}\int_{\tau}^{w_1} \frac{h(\boldsymbol{w})}{\sqrt{i(w_1-\tau)}(i(w_2-\tau))^{\frac{3}{2}}} dw_2 dw_1. $$

Sign lemma

The key ingredient:

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Lemma 9

Let $\ell_1, \ell_2 \in \mathbb{R}$, $\kappa \in \mathbb{R}$, *with* $(\ell_1, \ell_2 + \kappa\ell_1) \neq (0,0)$. *Then*

$$\begin{aligned}\operatorname{sgn}(\ell_1)\operatorname{sgn}(\ell_2+\kappa\ell_1)q^{\frac{\ell_1^2}{2}+\frac{\ell_2^2}{2}}&\\ = \int_\tau^{\tau+i\infty} \frac{\ell_1 e^{\pi i \ell_1^2 w_1}}{\sqrt{i(w_1-\tau)}} \int_\tau^{w_1} \frac{\ell_2 e^{\pi i \ell_2^2 w_2}}{\sqrt{i(w_2-\tau)}} dw_2 dw_1&\\ + \int_\tau^{\tau+i\infty} \frac{m_1 e^{\pi i m_1^2 w_1}}{\sqrt{i(w_1-\tau)}} \int_\tau^{w_1} \frac{m_2 e^{\pi i m_2^2 w_2}}{\sqrt{i(w_2-\tau)}} dw_2 dw_1&\\ + \frac{2}{\pi}\arctan(\kappa) q^{\frac{\ell_1^2}{2}+\frac{\ell_2^2}{2}},&\end{aligned}$$

where $m_1 := \frac{\ell_2+\kappa\ell_1}{\sqrt{1+\kappa^2}}$ *and* $m_2 := \frac{\ell_1-\kappa\ell_2}{\sqrt{1+\kappa^2}}$.

Completion

Completion:

Define

$$\widehat{\Psi}(\tau, w) := \frac{\sqrt{3}}{2\pi} \int_{\tau}^{w} {}_{*}\!\int_{\tau}^{w_1} \frac{h(\boldsymbol{w})}{\sqrt{i(w_1-\tau)}\left(i(w_2-\tau)\right)^{\frac{3}{2}}} dw_2 dw_1$$

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Note

$$\Psi(\tau) = \lim_{w\to\tau+i\infty} \widehat{\Psi}(\tau, w).$$

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Note

$$\Psi(\tau) = \lim_{w \to \tau + i\infty} \widehat{\Psi}(\tau, w).$$

Proposition 10

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\widehat{\Psi}\left(\frac{a\tau + b}{c\tau + d}, \frac{aw + b}{cw + d}\right) = \nu_\eta(M)^8 (c\tau + d)^2 \widehat{\Psi}(\tau, w).$$

B_2 parafermionic character

Let

$$F(\zeta) := \frac{1}{\left(\zeta_1 q, \zeta_1^{-1} q, \zeta_2 q, \zeta_2^{-1} q, \zeta_1\zeta_2 q, \zeta_1^{-1}\zeta_2^{-1} q, \zeta_1^2\zeta_2 q, \zeta_1^{-2}\zeta_2^{-1} q; q\right)_\infty}.$$

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Relation to Jacobi forms:

$$F(\zeta) = q^{\frac{1}{3}}\eta^4 \frac{\zeta_1^{-2}\zeta_2^{-\frac{3}{2}}(1-\zeta_1)(1-\zeta_2)(1-\zeta_1\zeta_2)\left(1-\zeta_1^2\zeta_2\right)}{\vartheta(z_1)\vartheta(z_2)\vartheta(z_1+z_2)\vartheta(2z_1+z_2)}.$$

Constant term

Define

$$\Phi(\tau) := \Phi_1(\tau) + \Phi_2(\tau),$$

Constant term

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$$\Phi(\tau) := \Phi_1(\tau) + \Phi_2(\tau),$$

where, with $Q_B(\boldsymbol{n}) := \frac{3}{2}n_1^2 + 3n_1n_2 + 3n_2^2$

$$\Phi_1(\tau) := \sum_{\boldsymbol{n}\in\mathbb{Z}^2+\left(\frac{1}{3},\frac{1}{6}\right)} (-1)^{n_1-\frac{1}{3}} \left(\operatorname{sgn}(n_2) + \operatorname{sgn}(n_1+n_2)\right) \operatorname{sgn}(n_1) \\ \times \left((n_1+2n_2)^2 - \frac{E_2(\tau)}{18}\right) q^{Q_B(\boldsymbol{n})},$$

$$\Phi_2(\tau) := \sum_{\boldsymbol{n}\in\mathbb{Z}^2+\left(\frac{1}{3},\frac{1}{6}\right)} (-1)^{n_1-\frac{1}{3}} \operatorname{sgn}(n_1+n_2) \operatorname{sgn}(n_2) n_1(n_1+2n_2) q^{Q_B(\boldsymbol{n})}.$$

Constant term

For $\boldsymbol{a} \in \mathbb{Z}^2 \setminus \{(0,0)\}$, let

$$\Lambda_{\boldsymbol{a}}(\tau) := \sum_{\boldsymbol{n} \in \mathbb{Z}^2 + \left(\frac{1}{3}, \frac{a_1}{2} + \frac{1}{6}\right)} (-1)^{(a_2+1)\left(n_1 - \frac{1}{3}\right)} \operatorname{sgn}(n_1) \operatorname{sgn}(n_1 + 2n_2) q^{Q_B(\boldsymbol{n})}.$$

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Moreover

$$\begin{aligned} F_0(\tau) := {} & \frac{E_2(\tau)+2}{4} + \frac{\eta(\tau)^6}{\vartheta\left(\frac{1}{2};\tau\right)^2} + 6q^{-\frac{1}{24}}\mathcal{D}_{\frac{1}{2}}(\omega_1(\tau)) - 6q^{-\frac{3}{8}}\mathcal{D}_{\frac{1}{2}}(\omega_0(\tau)) \\ & + q^{-\frac{1}{12}}\left(6\mathcal{D}_{\frac{1}{2}} - \frac{\eta(\tau)^6}{\vartheta\left(\frac{1}{2};\tau\right)^2} + q^{-\frac{1}{2}}\frac{\eta(\tau)^6}{\vartheta\left(\frac{\tau}{2};\tau\right)^2} - q^{-\frac{1}{2}}\frac{\eta(\tau)^6}{\vartheta\left(\frac{\tau+1}{2};\tau\right)^2}\right)(\phi_1(\tau)) \\ & - q^{-\frac{1}{3}}\left(6\mathcal{D}_{\frac{1}{2}} + \frac{\eta(\tau)^6}{\vartheta\left(\frac{1}{2};\tau\right)^2} + \frac{\eta(\tau)^6}{\vartheta\left(\frac{\tau}{2};\tau\right)^2} + \frac{\eta(\tau)^6}{\vartheta\left(\frac{\tau+1}{2};\tau\right)^2}\right)(\phi_2(\tau)), \end{aligned}$$

Constant term

with

$$\phi_r(\tau) := \sum_{n\in\mathbb{Z}+\frac{r}{6}} \operatorname{sgn}(n) q^{3n^2} \quad \text{and} \quad \omega_r(\tau) := \sum_{n\in\mathbb{Z}+\frac{r}{3}+\frac{1}{2}} (-1)^{n-\frac{r}{3}-\frac{1}{2}} \operatorname{sgn}(n) q^{\frac{3n^2}{2}}.$$

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Serre derivative: $\mathcal{D}_k := \frac{1}{2\pi i}\frac{\partial}{\partial\tau} - \frac{k}{12}E_2(\tau)$

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Theorem 11 (B.-Kaszian-Milas-Nazaroglu)

We have for $|q| < |\zeta_1^2|, |\zeta_2|, |\zeta_1^2\zeta_2| < 1$

$$\mathrm{CT}_{[\zeta]}\, F(\zeta) = \frac{q^{\frac{1}{3}}}{\eta(\tau)^8} F_0(\tau) + \frac{9q^{-\frac{1}{12}}}{2\eta(\tau)^8}\Phi(\tau) + \frac{q^{-\frac{1}{12}}}{\eta(\tau)^2}\left(\frac{\Lambda_{0,1}(\tau)}{\vartheta\left(\frac{1}{2};\tau\right)^2} + \frac{q^{-\frac{1}{4}}\Lambda_{1,0}(\tau)}{\vartheta\left(\frac{\tau}{2};\tau\right)^2} - \frac{q^{-\frac{1}{4}}\Lambda_{1,1}(\tau)}{\vartheta\left(\frac{\tau+1}{2};\tau\right)^2}\right).$$

Constant term

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Very painful!!!

Regularized integrals

Define

$$f_0(\boldsymbol{w}) := \vartheta^{[1]}_{3,1}(w_1)\vartheta^{[1]}_{3,2}(w_2) - \vartheta^{[1]}_{3,2}(w_1)\vartheta^{[1]}_{3,1}(w_2),$$
$$g_0(\boldsymbol{w}) := \vartheta^{[1]}_{\frac{3}{2},1}(w_1)\vartheta^{[1]}_{\frac{3}{2},0}(w_2) - \vartheta^{[1]}_{\frac{3}{2},0}(w_1)\vartheta^{[1]}_{\frac{3}{2},1}(w_2).$$

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$$g_0(\boldsymbol{w}) := \vartheta^{[1]}_{\frac{3}{2},1}(w_1)\vartheta^{[1]}_{\frac{3}{2},0}(w_2) - \vartheta^{[1]}_{\frac{3}{2},0}(w_1)\vartheta^{[1]}_{\frac{3}{2},1}(w_2).$$

Lemma 12

We have

$$\Phi(\tau) = \frac{1}{\pi}\int_\tau^{\tau+i\infty} {}_*\!\int_\tau^{w_1} \frac{(4f_0(\boldsymbol{w}) + g_0(\boldsymbol{w}))}{\sqrt{i(w_1-\tau)}(i(w_2-\tau))^{\frac{3}{2}}} \times \frac{\left(1 - \frac{\pi i}{6}(w_2-\tau)E_2(\tau)\right)}{\sqrt{i(w_1-\tau)}(i(w_2-\tau))^{\frac{3}{2}}} dw_2 dw_1.$$

Completion

Completion:

$$\widehat{\Phi}(\tau, w) := \frac{1}{\pi} \int_{\tau}^{w} {}_{*}\int_{\tau}^{w_1} \frac{\left(4f_0(\boldsymbol{w}) + g_0(\boldsymbol{w})\right)\left(1 - \frac{\pi i}{6}(w_2 - \tau)E_2(\tau)\right)}{\sqrt{i(w_1 - \tau}(i(w_2 - \tau)^{\frac{3}{2}}} dw_2 dw_1$$

Completion

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$$\widehat{\Phi}(\tau, w) := \frac{1}{\pi}\int_{\tau}^{w} {}_{*}\int_{\tau}^{w_1} \frac{(4f_0(\boldsymbol{w}) + g_0(\boldsymbol{w}))\left(1 - \frac{\pi i}{6}(w_2-\tau)E_2(\tau)\right)}{\sqrt{i(w_1-\tau}(i(w_2-\tau)^{\frac{3}{2}}} dw_2 dw_1$$

Proposition 13

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ *we have*

$$\widehat{\Phi}\left(\frac{a\tau+b}{c\tau+d}, \frac{aw+b}{cw+d}\right) = \nu_\eta(M)^{10}(c\tau+d)^3\,\widehat{\Phi}(\tau, w).$$

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- Build a theory of Fourier coefficients of multi-variable Jacobi forms.

Happy Birthday Steve