# Semidefinite Relaxations of Products of Nonnegative Forms

Workshop on Real Algebraic Geometry and Algorithms for Geometric Constraint Systems

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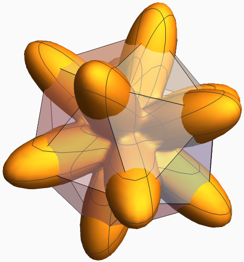

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$$\Downarrow$$

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Provable approximation guarantees

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- Compact representation: represented in $O(n^2 d)$ space
- Polynomial time solution when $d$ is fixed, NP-hard when $d = \Omega(n)$

# Applications and Motivation

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$$r(M) \coloneqq \max_{\|x\|=1, x\in\mathbb{C}^n} \prod_{i=1}^{n} |\langle x, v_i\rangle|^2, \quad \frac{n!}{n^n} r(M) \le \mathrm{per}(M)$$

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And more! (Solving systems of quadratic equations, linear polarization constants, Nash social welfare ...)

# Our Contributions

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Compute: SDP with $O(\binom{n+d}{d})$ vars/consts   Approx: $\Omega(\frac{1}{n})$

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- Introduce higher-degree relaxations that trade off computation with approximation quality
- Exhibit integrality gap instances that show our analysis of our SDP based relaxation is tight

# Semidefinite Relaxation

$$\max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x \rangle^{1/d} \qquad \text{(Opt)}$$

rank-1 relax ↓

$$\max_{X} \prod_{i=1}^{d} \langle X, A_i \rangle^{1/d} \quad \text{s.t.} \quad \mathrm{Tr}(X) = 1,\ X \succeq 0$$

use AM-GM ↓

$$\min_{\lambda,\alpha} \lambda \quad \text{s.t.} \quad \frac{1}{d}\sum_{i=1}^{d} \alpha_i A_i \preceq \lambda I, \quad \prod_i \alpha_i \geq 1,\ \alpha_i > 0 \qquad \text{(OptSDP)}$$

dual: the two problems above are connected by ⟷

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**Theorem**
*Let* $r = \operatorname{rank}(X^*) \leq n$, *$\gamma$ be Euler's constant, $\phi$ be digamma function*

$$c_r(\mathbb{K})\, \textsc{OptSDP} \leq \textsc{Opt} \leq \textsc{OptSDP}$$

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$$c_r(\mathbb{K}) = \begin{cases} \exp(-\gamma - \log 2 - \phi\left(\frac{r}{2}\right) + \log\left(\frac{r}{2}\right)) > 0.2807 & \textit{if } \mathbb{K} = \mathbb{R} \\ \exp(-\gamma - \phi(r) + \log(r)) > 0.5614 & \textit{if } \mathbb{K} = \mathbb{C} \end{cases}$$

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Lower bound expected value of objective:

$$\begin{aligned}
\textsc{Opt} &\geq \mathbb{E}\left[\prod_{i=1}^{d} \langle \hat{x}, A_i \hat{x} \rangle^{1/d}\right] \\
&= \mathbb{E}\left[\exp\left(\frac{1}{d}\sum_{i=1}^{d} \log \langle \hat{x}, A_i \hat{x} \rangle\right)\right] \\
&\geq \exp\left(\frac{1}{d}\sum_{i=1}^{d} \mathbb{E}[\log \langle \hat{x}, A_i \hat{x} \rangle]\right) \\
&\geq c_r(\mathbb{K})\textsc{OptSDP}
\end{aligned}$$

# Application: convex hull of image of quadratic map

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**Theorem**
*Let $a \in \mathrm{conv}(\varphi(\mathbb{K}^n)) \cap \Delta_d$. Then there exists a point $b \in \varphi(\mathbb{K}^n) \cap \Delta_d$ such that*

$$D(a \parallel b) = \sum_{i=1}^{d} a_i \ln\left(\frac{a_i}{b_i}\right) \le \log(c_r(\mathbb{K}))$$

Proved by Barvinok (2014) for a larger constant, our analysis gives asymptotically optimal constant

## Higher-order Relaxations

$\textsc{OptSDP}$ constructed using AM/GM inequality. If $\prod_i \alpha_i = 1$,

$$\prod_{i=1}^{d} \langle x, A_i x\rangle^{1/d} \leq x^T \left( \frac{1}{d} \sum_{i=1}^{d} \alpha_i A_i \right) x$$

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Let $E_k$ be elementary symmetric polynomials:

$$E_k(y_1, \ldots, y_d) = \binom{d}{k}^{-1} \sum_{I \subseteq [d], |I| = k} \prod_{i \in I} x_i$$

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Use Sum-of-Squares to construct relaxations $\textsc{OptSoS}_k$ for $1 \leq k \leq d$

$$\textsc{Opt} \leq \textsc{OptSDP} = \textsc{OptSoS}_1 \leq \textsc{OptSoS}_d$$

Trades off computation for accuracy

# Example: Icosahedral form

Let $\psi$ be golden ratio, $C$ chosen so that $\max_{x^2+y^2+z^2=1} p(x,y,z) = 1$.

$$p(x,y,z) = C\left[(x+\psi y)(x-\psi y)(y+\psi z)(y-\psi z)(z+\psi x)(z-\psi x)\right]^2$$

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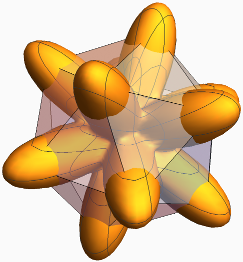

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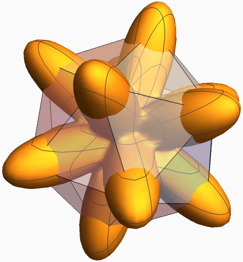

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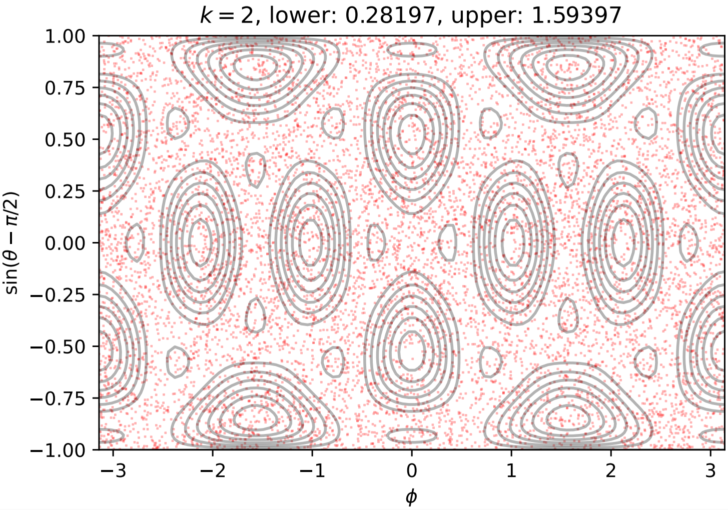
We implement a randomized rounding algorithm to obtain feasible solution from relaxations $\textsc{OptSoS}_k$

# Distribution sampled from rounding algorithm



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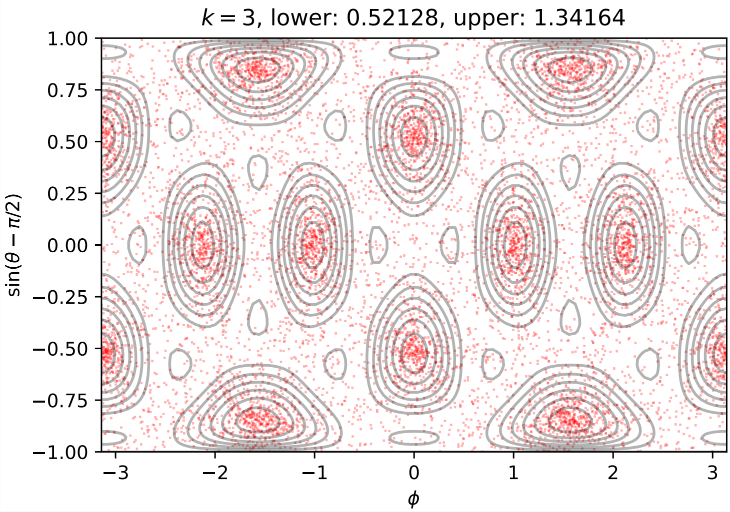

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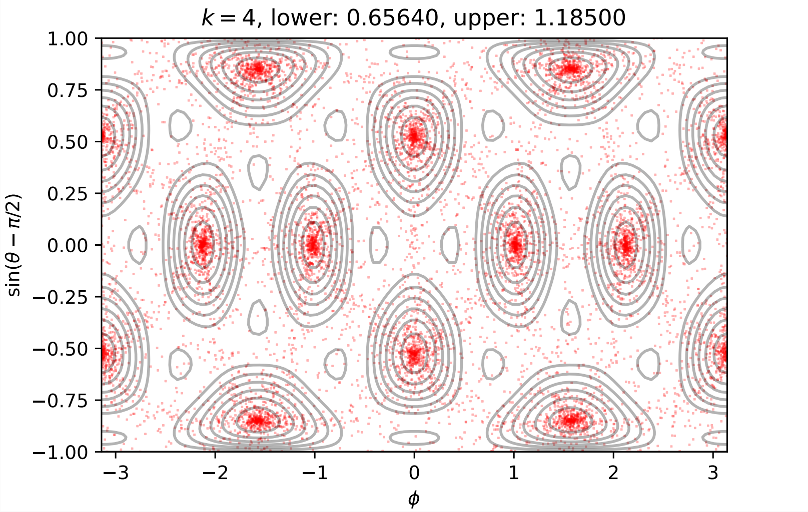

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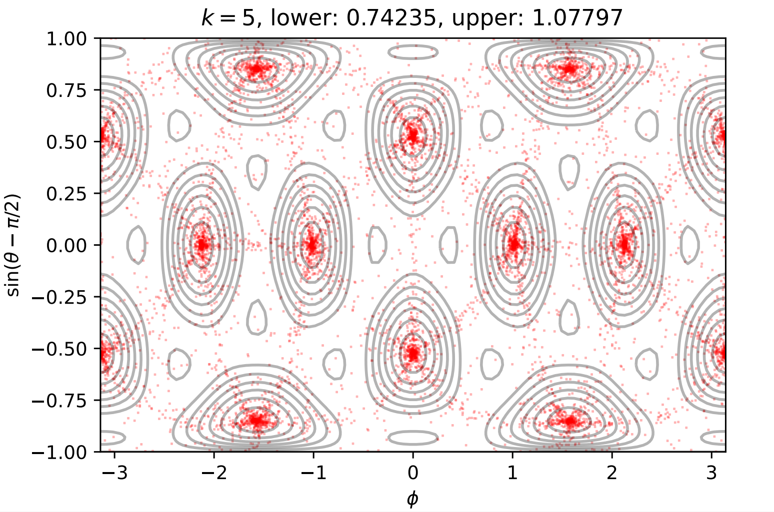

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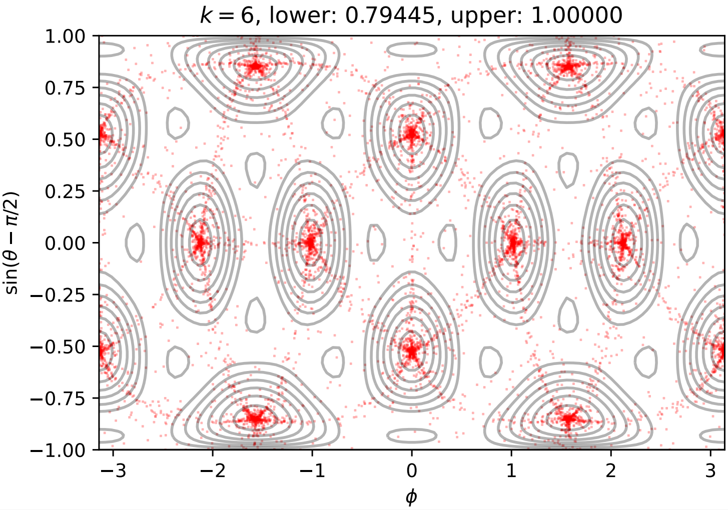

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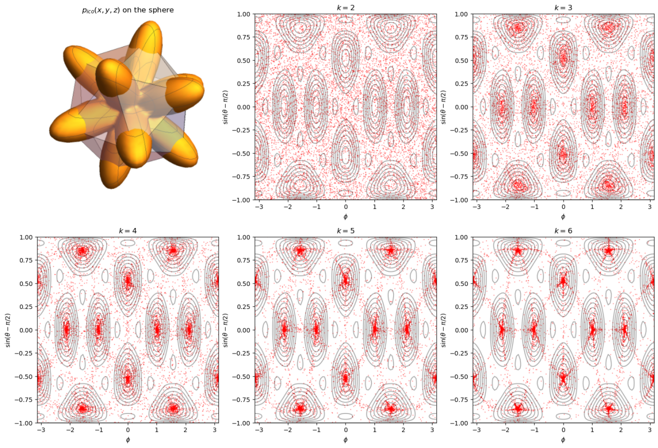


Distribution concentrates towards optima as $k$ increases

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Future work:

- Low-rank guarantees of solution from symmetry
- How to generate intermediate Sum-of-Squares relaxations for other high degree polynomial optimization problems?