Snappability and singularity-distance of frameworks

Georg Nawratil[1,2]

[1]Institute of Discrete Mathematics and Geometry, TU Wien
www.dmg.tuwien.ac.at/nawratil/

[2]Center for Geometry and Computational Design, TU Wien

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Outline

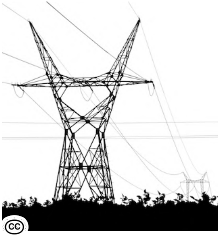
1. Introduction
2. Theory
3. Examples

1. Introduction

For the detailed references of the cited literature please see

GN: *Snappability and singularity-distance of pin-jointed body-bar frameworks*. arXiv:2101.02490 (2021)

Frameworks



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 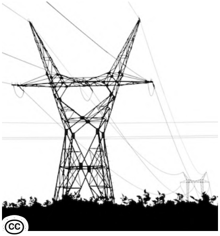 

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 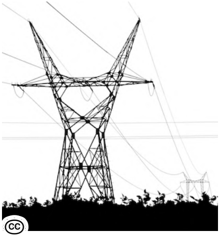 

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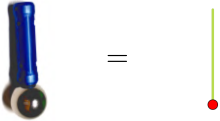
Geometric abstraction

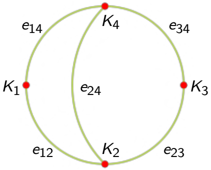
Knots are reduced to points and bars to straight line-segments.



Fundamentals

Graph G of a framework

consists of a knot-set $\mathcal{K} = \{K_1, \ldots, K_s\}$, where knots K_i and K_j are connected by edges e_{ij} ($\Rightarrow$ combinatorial structure).



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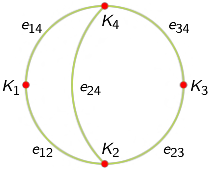
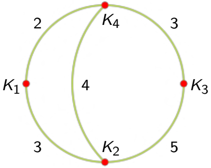
Inner geometry

is determined by assigning to each edge e_{ij} a length $L_{ij} > 0$ ($\Leftrightarrow$ fixing intrinsic metric).





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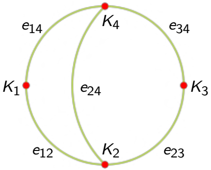

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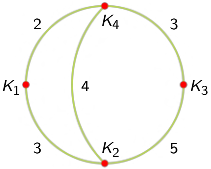
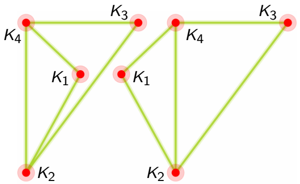


Realization $G(\mathbf{K})$

with $\mathbf{K} = (\mathbf{k}_1, \ldots, \mathbf{k}_s) \in \mathbb{R}^{sd}$ corresponds to the embedding of the framework with fixed inner geometry into the Euclidean d-space.



Algebraic approach to rigidity theory

The relation that two knots K_i and K_j are edge-connected can also be expressed algebraically as $\|\mathbf{k}_i - \mathbf{k}_j\|^2 = L_{ij}^2$.

In addition we can add 6 (for $d = 3$) or 3 (for $d = 2$) linear conditions to eliminate isometries. We end up with n algebraic conditions in $m = sd$ unknowns constituting an algebraic variety $A(c_1, \ldots, c_n)$.

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Def.: *A realization is* **flexible**

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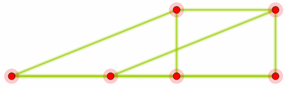
Def.: *A realization is* **rigid**

if it corresponds to an real isolated solution of $A(c_1, \ldots, c_n)$. If it is unique then we have a **global** rigidity; otherwise a **local** one.

Algebraic approach to rigidity theory

Example: planar parallel mechanism



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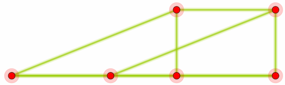

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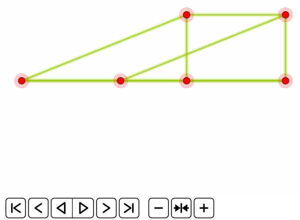

∃ paradox mobile realization

Rigid realization

The realization is called **isostatic** (minimally rigid) if the removal of any edge constraint will make the realization flexible ($\Leftrightarrow m = n$).

Remark: There is also a combinatorial characterization of isostaticity for generic frameworks in $\mathbb{R}^2$ according to **Laman** (1970).

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We can compute in a realization the tangent-hyperplane to each of the hypersurfaces $c_i = 0$ in $\mathbb{R}^m$ for $i = 1, \ldots, n$. The normal vectors of these tangent-hyperplanes constitute the columns of the $m \times n$ **rigidity matrix** $\mathbf{R}_{G(\mathbf{K})}$ of the realization $G(\mathbf{K})$.

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For $rk(\mathbf{R}_{G(\mathbf{K})}) < m$ the realization $G(\mathbf{K})$ is **infinitesimal flexible**; i.e. the hyperplanes have a positive-dimensional affine subspace in common. Therefore the intersection multiplicity of the n hypersurfaces is at least two in a shaky realization.

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Remark: For isostatic frameworks the infinitesimal flexibility is characterized by $\det(\mathbf{R}_{G(\mathbf{K})}) = 0$.

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Open problem: The meaning of closeness!

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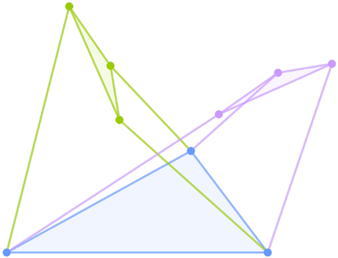
Connection between shakiness and snapping



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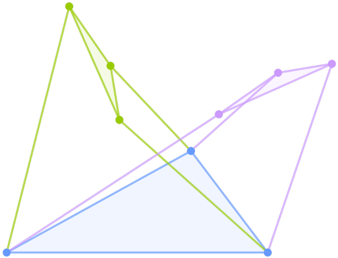

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From two realizations with **the same inner geometry** we get a **shaky realization** with a different intrinsic metric.

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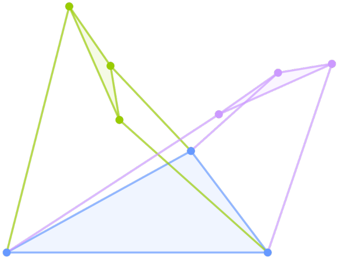

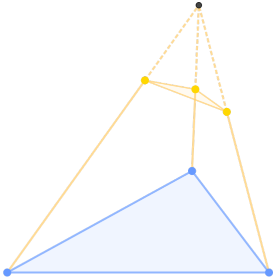

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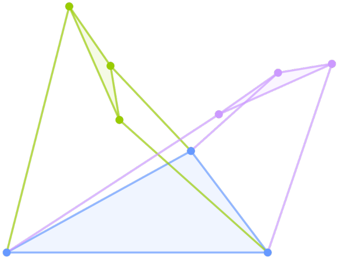

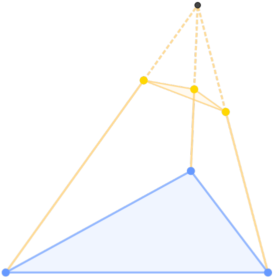

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“Deaveraging”

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Gorkavyy & Fesenko (2019) studied how slight variations on the SD's edge lengths produce significant shape variations. They suggested estimates to quantify these intrinsic and extrinsic variations.

Holmes-Cerfon, Theran & Gortler (2021) computed bounds for these quantities for arbitrary bar-joint frameworks.

2. Theory

In the presentation we restrict to pin-jointed frameworks composed of bars and triangular panels but the theory can be generalized to polygonal panels and polyhedra as well; cf.

GN: *Snappability and singularity-distance of pin-jointed body-bar frameworks*. arXiv:2101.02490 (2021)

Physical Model of Deformation

Due to the fact that the elastic deformation during the process of snapping are expected to be small, we can apply **Hooke's law**; i.e.

$$\underbrace{\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}}_{\mathbf{e}} = \underbrace{\frac{1}{E}\begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{pmatrix}}_{=:\mathbf{D}(\nu)} \begin{pmatrix} \delta_x \\ \delta_y \\ \tau_{xy} \end{pmatrix}. \qquad (1)$$

- normal stress $\delta_{x/y}$ and normal strain $\varepsilon_{x/y}$ in x/y-direction
- shear stress τ_{xy} and shear strain γ_{xy} in the xy-plane
- Poisson ration ν and Young modulus E

For a bar (in x-direction) the relation reduces to $\varepsilon_x = \frac{\delta_x}{E}$.

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(II) all bars have the same cross-sectional area A,

(III) triangular bar structure and triangular panel are made of the same amount of material.

Strain energy according to Green-Lagrange

The deformation of the triangular panel K_i, K_j, K_k into K'_i, K'_j, K'_k can be represented by a 2×2 matrix $\mathbf{F}$ in terms of edge lengths.

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Based on $\mathbf{F}$ the Green-Lagrange (GL) strains can be computed as

$$\begin{pmatrix} \varepsilon_x & \gamma_{xy} \\ \gamma_{xy} & \varepsilon_y \end{pmatrix} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right).$$

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The elastic GL strain energy of the deformation is calculated as

$$U_{ijk} = V_{ijk}\frac{1}{2}\mathbf{e}^T\mathbf{D}^{-1}\mathbf{e} \quad \text{where } V_{ijk} \text{ denotes the panel volume.}$$

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Analogue: GL strain energy of a deformed bar can be computed as

$$U_{ij} = \tfrac{EA}{8L_{ij}^3}({L'_{ij}}^2 - L_{ij}^2)^2 \quad \text{where } L'_{ij} \text{ denotes the deformed length.}$$

Total elastic strain energy density

$$u(\mathbf{L}') = \frac{\sum U_{ij} + \sum U_{ijk}}{\sum AL_{ij} + \sum A(L_{ij} + L_{ik} + L_{jk})} \quad \text{with } \mathbf{L}' = (\ldots, L'_{ij}, \ldots)$$

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Lemma 1.

$u(\mathbf{L}')$ is a fourth order polynomial with respect to the variables L'_{ij} which only appear with even powers, but it does not depend on A. Moreover, $u(\mathbf{L}')$ is positive semi-definite.

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Remark. Therefore $u(\mathbf{L}')$ can be written in matrix formulation as $u(\mathbf{Q}') = \mathbf{Q}'^T \mathbf{M} \mathbf{Q}'$ where $\mathbf{M}$ is a symmetric $(b+1)$-matrix and $\mathbf{Q}' := (1, \ldots, Q'_{ij}, \ldots)^T$ is composed of the b squared edge lengths $Q'_{ij} := L'^2_{ij}$ and the number 1.

Pseudometric on the space $\mathbb{R}^b$

Lemma 2.

The following function

$$d : \mathbb{R}^b \times \mathbb{R}^b \to \mathbb{R}_{\geq 0} \quad \text{with} \quad (\mathbf{L}', \mathbf{L}'') \mapsto d(\mathbf{L}', \mathbf{L}'') := \frac{|u(\mathbf{L}') - u(\mathbf{L}'')|}{E}$$

is a pseudometric on the b-dimensional space of intrinsic framework metrics given by $\mathbf{L}'$ and $\mathbf{L}''$, respectively. Moreover, the pseudometric does not depend on the choice of E.

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Proof: One can easily check the axioms for a pseudometric:

(1) $d(\mathbf{L}', \mathbf{L}'') \geq 0$ (2) $d(\mathbf{L}', \mathbf{L}') = 0$

(3) $d(\mathbf{L}', \mathbf{L}'') = d(\mathbf{L}'', \mathbf{L}')$ (4) $d(\mathbf{L}', \mathbf{L}''') \leq d(\mathbf{L}', \mathbf{L}'') + d(\mathbf{L}'', \mathbf{L}''')$

Due to Assumption I, Young's modulus E factors out of $u(\mathbf{L}')$. □

$u(\mathbf{L}') \implies u(\mathbf{K}')$ and its critical points

Theorem 1.

The critical points of the total elastic stain energy density $u(\mathbf{K}')$ of an isostatic framework correspond to realizations $G(\mathbf{K}')$ that are either undeformed or deformed and shaky.

$u(\mathbf{L}') \Longrightarrow u(\mathbf{K}')$ and its critical points

Theorem 1.

The critical points of the total elastic stain energy density $u(\mathbf{K}')$ of an isostatic framework correspond to realizations $G(\mathbf{K}')$ that are either undeformed or deformed and shaky.

Proof: Based on characterization of shakiness in terms of self-stress: If one can assign to each edge e_{ij} of $G(\mathbf{k}')$ a *stress* $\omega_{ij} \in \mathbb{R}$ in a way that for each knot the so-called *equilibrium condition*

$$\sum_{i<j} \omega_{ij}(\mathbf{k}'_i - \mathbf{k}'_j) + \sum_{i>j} \omega_{ji}(\mathbf{k}'_i - \mathbf{k}'_j) = \mathbf{o}$$

is fulfilled, then $\omega = (\dots, \omega_{ij}, \dots) \in \mathbb{R}^b$ is referred as *self-stress*. If $\omega \neq \mathbf{o}$, then the realization $G(\mathbf{k}')$ of an isostatic framework is shaky.

$u(\mathbf{L}') \implies u(\mathbf{K}')$ and its critical points

The system of equations characterizing critical points of $u(\mathbf{K}')$:

$$\nabla_i\, u(\mathbf{K}') = \mathbf{o} \quad \text{with} \quad \nabla_i\, u(\mathbf{K}') = \left(\frac{\partial u}{\partial k'_{i,1}}, \ldots, \frac{\partial u}{\partial k'_{i,d}} \right) \quad i = 1, \ldots, s$$

where $(k'_{i,1}, \ldots, k'_{i,d})$ is the coordinate vectorof $\mathbf{k}'_i \in \mathbb{R}^d$.

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Due to the sum rule for derivatives we only have to investigate ∇_i of $U_{ijk}(\mathbf{K}')$ and $U_{ij}(\mathbf{K}')$, which can be written as

$$\nabla_i\, U_{ij} = \omega_{ij}(\mathbf{k}'_i - \mathbf{k}'_j) \qquad \text{with} \quad \omega_{ij} = \frac{A(L'^2_{ij} - L^2_{ij})}{2L^3_{ij}}$$

$$\nabla_i\, U_{ijk} = \omega_{ij}(\mathbf{k}'_i - \mathbf{k}'_j) + \omega_{ik}(\mathbf{k}'_i - \mathbf{k}'_k) \quad \text{with} \quad \omega_{ij} = \ldots, \quad \omega_{ik} = \ldots$$

Therefore $\nabla_i\, u(\mathbf{K}')$ has the shape of the equilibrium condition. □

Further connection between shakiness and snapping

Def.: *A realization is* **stable**

if it corresponds to a local minimum of the total elastic strain energy (density) of the framework.

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If an isostatic framework snaps out of a stable realization $G(\mathbf{K})$ by applying the minimum GL strain energy needed to it, then the corresponding deformation has to pass a shaky realization $G(\mathbf{K}')$ at the maximum state of deformation.

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Proof: We think of u as a graph function over the space $\mathbb{R}^{sd}$ of knot configurations. In order to get out of the valley of the local minimum $(\mathbf{K}, u(\mathbf{K}))$ with a minimum of energy needed, one has to pass a *saddle point* $(\mathbf{K}', u(\mathbf{K}'))$. □

Snappability

Def.: Snappability *of a realization* $G(\mathbf{K})$

is given by $s(\mathbf{K}) := d(\mathbf{L}, \mathbf{L}')$ with $G(\mathbf{K}')$ of Theorem 2.

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(1) We compute the set $\mathcal{S}$ of saddle points. Let us assume that $G(\mathbf{K}') \in \mathcal{S}$ yields the minimal value for $d(\mathbf{L}, \mathbf{L}')$.

(2) $\mathbf{Q}_t := \mathbf{Q} + t(\mathbf{Q}' - \mathbf{Q})$ with $t \in [0, 1]$ implies a path $\mathbf{L}_t$ in $\mathbb{R}^b$.
Along this path the deformation energy of each bar and triangular plate is *monotonic increasing* ensuring that the minimum mechanical work needed is applied to reach $G(\mathbf{K}')$.
This results from Lemma 1, as $U_{ijk}(\mathbf{L}_t)$ as well as $U_{ij}(\mathbf{L}_t)$ are quadratic functions in t, which are at their minima for $t = 0$.

Snappability

(3) The path $\mathbf{L}_t$ corresponds to different 1-parametric deformations of realizations in $\mathbb{R}^d$. If among these a deformation $G(\mathbf{K}_t)$ with

$$G(\mathbf{K}_t)\big|_{t=0} = G(\mathbf{K}), \quad G(\mathbf{K}_t)\big|_{t=1} = G(\mathbf{K}')$$

exists, then $G(\mathbf{K})$ is deformed into $G(\mathbf{K}')$ under $\mathbf{L}_t \Longrightarrow s(\mathbf{K})$

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Remark: Computationally the property of step (3) can be checked by a user defined homotopy approach (e.g. software Bertini).

(4) Otherwise we redefine $\mathcal{S}$ as $\mathcal{S} \setminus \{G(\mathbf{K}')\}$ and run again the procedure. If we end up with $\mathcal{S} = \varnothing$ then we set $s(\mathbf{K}) = \infty$.

Singularity-distance

Theorem 3.

For a non-shaky realization $G(\mathbf{K})$ of an isostatic framework the singularity-distance $\varsigma(\mathbf{K})$ equals the snappability $s(\mathbf{K})$.

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Proof: $\varsigma(\mathbf{K}) \leq s(\mathbf{K})$ has to hold, as $G(\mathbf{K}')$ of Theorem 2 is shaky. We show that the assumption $\varsigma(\mathbf{K}) < s(\mathbf{K})$ implies a contradiction.

We denote by $G(\mathbf{K}'')$ shaky configuration implying $\varsigma(\mathbf{K}) = d(\mathbf{L}, \mathbf{L}'')$.

Then $\mathbf{Q}_t := \mathbf{Q} + t(\mathbf{Q}'' - \mathbf{Q})$ with $t \in [0,1]$ corresponds to a set of 1-parametric deformations $\{G(\mathbf{K}_t^1), G(\mathbf{K}_t^2), \ldots\}$.

A subset $\mathcal{D}$ of this set has the property $G(\mathbf{K}_t^i)|_{t=1} = G(\mathbf{K}'')$ where $\#\mathcal{D} > 1$ holds as $G(\mathbf{K}'')$ is shaky. Hence the framework can snap out of $G(\mathbf{K})$ over $G(\mathbf{K}'')$ which contradicts $\varsigma(\mathbf{K}) < s(\mathbf{K})$. □

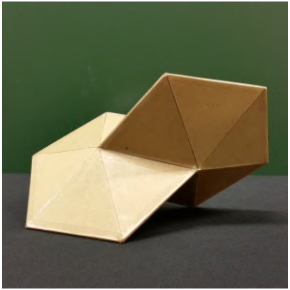
3. Examples

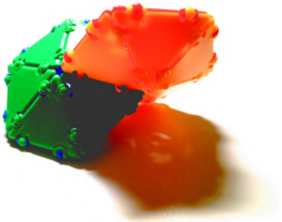
For a detailed comparison of the following presented results with those given in the literature please see

GN: *Snappability and singularity-distance of pin-jointed body-bar frameworks*. arXiv:2101.02490 (2021)

Siamese Dipyramid

Michael Goldberg (1978):
The polyhedron consists of 20 equilateral triangles and has 12 vertices and 30 edges.





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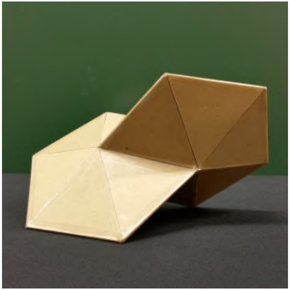

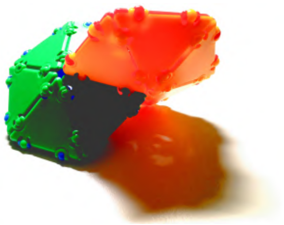

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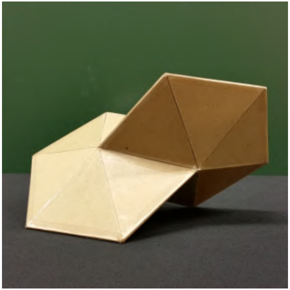
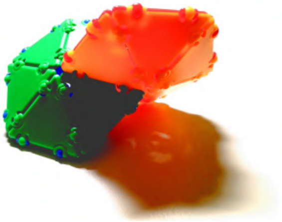
The SD has a reflexion-symmetry with respect to two orthogonal planes.

The deformation of the SD keeping this symmetry property is 11-dimensional.





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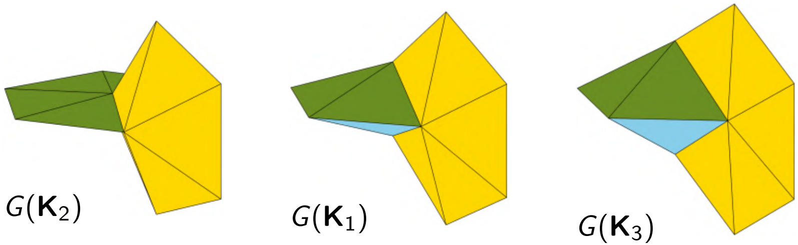
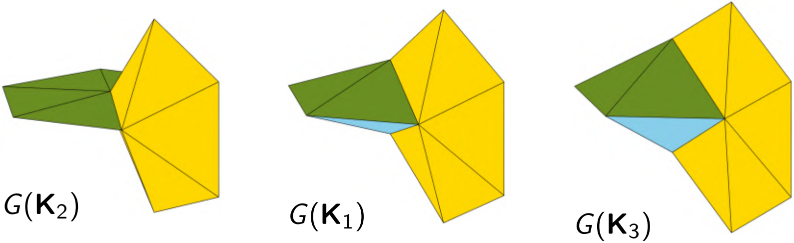


The SD can snap out of the symmetric realization $G(\mathbf{K}_1)$ into one of the two asymmetric realizations $G(\mathbf{K}_2)$ and $G(\mathbf{K}_3)$, respectively.

Siamese Dipyramid



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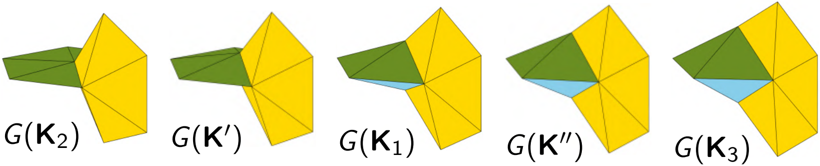
Isostaticity. Every closed polyhedral surface of genus 0 with triangular faces is isostatic. This isostaticity remains intact under the assumption of the 2-fold reflexion-symmetry.

Siamese Dipyramid

Results for joint-bar/panel-hinge framework

The obtained system of 11 equations ∇u results in 177 147 paths within a total degree homotopy. The path tracking done by the software Bertini ends up in 22 153/20 305 finite real solutions. After reduction to the set $\mathcal{S}$ we remain with 21 904/20 056 solutions.



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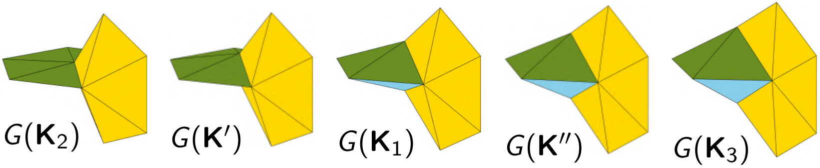

We get $s(\mathbf{K}_{1,2,3}) = \varsigma(\mathbf{K}_{1,2,3}) = 1.661376 \cdot 10^{-6}/4.466362 \cdot 10^{-6}$.

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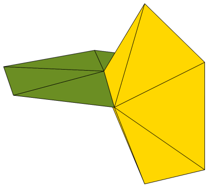

Model flexibility

The model snaps between three realizations.

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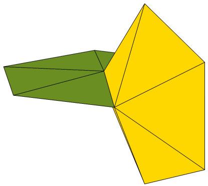

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The maximal change of an edge-length is approximately **3mm** if the triangles have a side length of **1m**.

Four-Horn

Casper Schwabe (1984):
The polyhedron has 10 vertices, 24 edges and consists of 16 congruent isosceles triangles with $\alpha := \measuredangle(\text{leg,base}) = 22.5^\circ$.

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The deformation of FH_α keeping this symmetry property is 9-dimensional.

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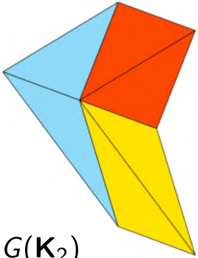
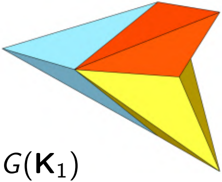
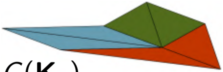


$G(\mathbf{K}_2)$



$G(\mathbf{K}_1)$



$G(\mathbf{K}_3)$

FH_α can snap out of the symmetric realization $G(\mathbf{K}_1)$ into one of the two flat realizations $G(\mathbf{K}_2)$ and $G(\mathbf{K}_3)$, which are **shaky**.

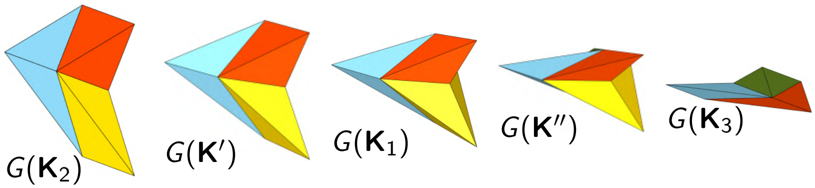
We consider FH_α where α equals 15°, 22.5° and 30°, respectively.

Four-Horn

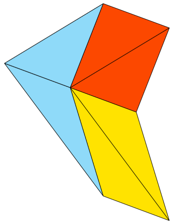
Results for joint-bar/panel-hinge framework

The obtained system of 9 equations ∇u results in 19 683 paths within a total degree homotopy. Note that $\varsigma(\mathbf{K}_{2,3}) = 0$ holds.

α	# real solutions	# $\mathcal{S}$	$s(\mathbf{K}_{1,2,3}) = \varsigma(\mathbf{K}_1)$
15°	923/1 324	897/1 238	$9.864008 \cdot 10^{-11}/6.288380 \cdot 10^{-8}$
22.5°	924/1 259	863/1 242	$1.753810 \cdot 10^{-8}/1.748173 \cdot 10^{-6}$
30°	917/1 457	819/1 360	$2.035395 \cdot 10^{-7}/2.340885 \cdot 10^{-5}$



Four-Horn



Model flexibility

The model snaps between three realizations.

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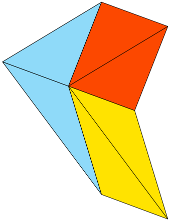


Model flexibility

The model snaps between three realizations.

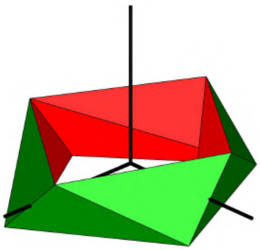
The maximal change of an edge-length is approximately

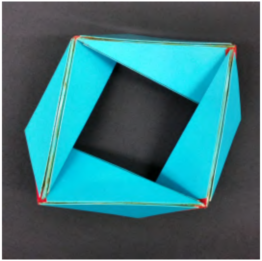
0.02mm. . . $\alpha = 15°$
0.29mm. . . $\alpha = 22.5°$
2.06mm. . . $\alpha = 30°$

if the average edge length equals **1m**.

4R Loop

Walter Wunderlich (1971):
Closed serial chain composed of four directly congruent tetrahedral chain elements, which are jointed by four hinges.





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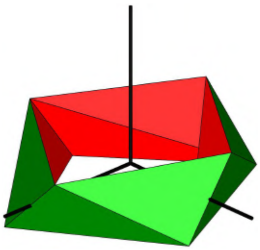
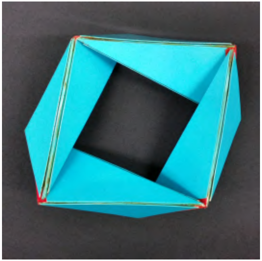
It consists of two sets of 8 congruent triangles and has 8 vertices and 20 edges.





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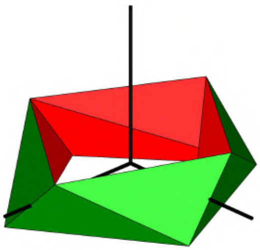
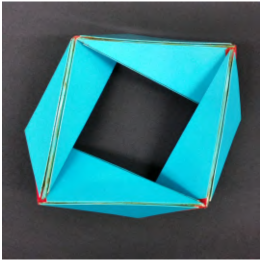
The 4R loop has a threefold reflexion symmetry with respect to three copunctal lines, which are pairwise orthogonal.





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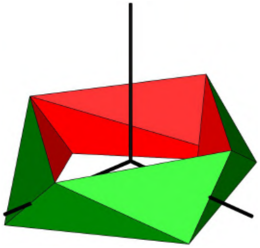
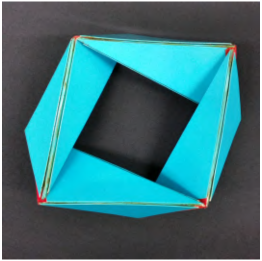
The 4R loop has a threefold reflexion symmetry with respect to three copunctal lines, which are pairwise orthogonal.

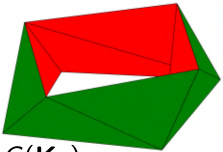
The deformation of the 4R loops keeping this symmetry property is 6-dimensional. Under this symmetry assumption the framework is also isostatic.



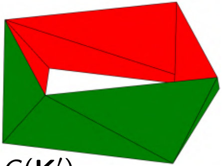


4R Loop

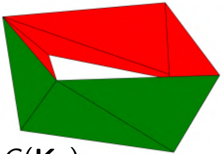
The 4R loop snaps out of the realization $G(\mathbf{K}_1)$ over the shaky configuration $G(\mathbf{K}')$ into the realization $G(\mathbf{K}_2)$.



$G(\mathbf{K}_1)$



$G(\mathbf{K}')$



$G(\mathbf{K}_2)$

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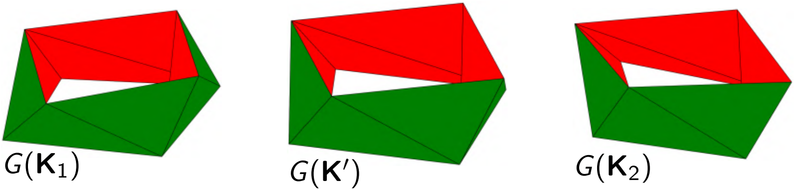
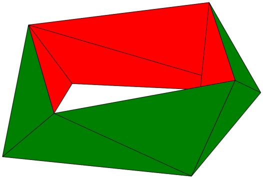


Results for joint-bar/panel-hinge framework

The obtained system of 6 equations ∇u results in 729 paths within a total degree homotopy.

# real solutions	# $\mathcal{S}$	$s(\mathbf{K}_{1,2}) = \varsigma(\mathbf{K}_{1,2})$
113/161	96/144	$6.762914 \cdot 10^{-7}/9.363722 \cdot 10^{-6}$

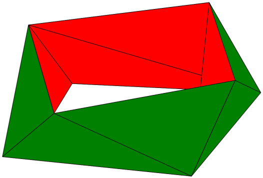
4R Loop



Model flexibility

The model snaps between two realizations.

4R Loop



Model flexibility

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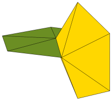
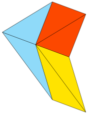
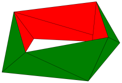
The maximal change of an edge-length is approximately **2.36mm** if the average edge length equals **1m**.

Acknowledgment

The author is supported by grant P 30855-N32 of the Austrian Science Fund FWF as well as by FWF project F77 (SFB "Advanced Computational Design", subproject SP7).







Thank you for your attention!