Noncommutative convexity

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Some historical developments in noncommutative convexity

A unital **operator system** is a closed unital self-adjoint subspace of a C*-algebra ($1 \in S = S^* \subseteq A$ for some A).

(Arveson 1969, 1972, 1998): Considers unital operator systems as noncommutative analogues of function systems (categorically dual to compact convex sets). Proves his extension theorem and conjectures existence of analogue of Choquet boundary for operator systems.

(Wittstock 1981): Introduces notion of matrix convex set.

(Effros-Winkler 1997): Establishes analogue of Hahn-Banach separation theorem for matrix convex sets.

(Webster-Winkler 1999): Establishes dual equivalence between category of unital operator systems and category of matrix convex sets.

(Arveson 2007, DK 2015): Proof of existence of Choquet boundary of an operator system.

Subsequently: Major developments in operator spaces/systems. Interesting examples of matrix convex sets arising from e.g. noncommutative real algebraic geometry (Helton-McCullough et al.). But also serious issues e.g. (provably) no good notion of extreme point for matrix convex sets.

Today: Discuss refined notion of nc convex set, existence of extreme points, categorical equivalence between nc convex sets and operator systems (both unital and non-unital case), nc function theory, noncommutative Choquet theory and some application, including to Drury-Arveson space.

Classical (dual) equivalence of categories:

Unital commutative C*-algebras with unital*-homomorphisms		Compact Hausdorff spaces with continuous maps
$\mathrm{C}(X)$	$\longleftrightarrow$	$X = \Omega(\mathrm{C}(X))$

A **unital function system** (archimedean order unit space) is a closed unital self-adjoint subspace of a commutative C*-algebra ($1 \in F = F^* \subseteq \mathrm{C}(X)$ for some X).

Theorem (Kadison 1951)

A unital function system F with state space K is unitally order isomorphic to the function system $A(C)$ of continuous affine functions on $C = S(F)$.

Function systems with unital order homomorphisms		Compact convex sets with continuous affine maps
$\mathrm{A}(C)$	$\longleftrightarrow$	$K = S(\mathrm{A}(C))$

Noncommutative convex sets

Let S be a unital operator system (i.e. $1 \in S = S^* \subseteq A$).

Definition

The **nc state space** of S is $K = \coprod_{n \leq \kappa} K_n$,

$$K_n = \{x : A \to M_n \text{ unital completely positive}\},$$

for a suitably large infinite cardinal κ (if S is separable then can take $\kappa = \aleph_0$).

Refines notion of matrix convex set where $n < \infty$. Subtle but extremely important difference.

Note: each K_n is compact in the point-weak* topology and K is closed under **nc convex combinations**:

$$\sum \alpha_i^* x_i \alpha_i \in K_n$$

for $x_i \in K_{n_i}$ and $\alpha_i \in M_{n,n_i}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$.

Definition (DK2019)

A **compact nc convex set** over a dual operator space E is a graded set $K = \coprod_{n \leq \kappa} K_n$ with $K_n \subseteq M_n(E)$ such that each K_n is compact in the dual topology on $M_n(E)$ and K is closed under nc convex combinations:

$$\sum \alpha_i^* x_i \alpha_i \in K_n$$

for $x_i \in K_{n_i}$ and $\alpha_i \in M_{n,n_i}$ satisfying $\sum \alpha_i^* \alpha_i = 1_n$.

Example

The **nc d-ball** $K = \coprod_{n \leq \aleph_0} K_n$, consisting of "row contractions,"

$$K_n = \{\alpha = (\alpha_1, \ldots, \alpha_d) \in M_n^d : \|(\alpha_1, \ldots, \alpha_d)\| \leq 1\}.$$

Let $\mathcal{O}_d = \mathrm{C}^*(v_1, \ldots, v_d)$ denote the Cuntz algebra. Then K is the nc state space of the Cuntz operator system

$$\mathrm{span}\{1, v_1, v_1^* \ldots, v_n, v_n^*\}.$$

Noncommutative functions

For a unital operator system S with nc state space $K = \coprod K_n$, an element $a \in S$ gives rise to a function $\hat{a} : K \to \coprod M_n$,

$$\hat{a}(x) = x(a), \quad x \in K.$$

The function $\hat{a}$ is graded, respects direct sums and is equivariant with respect to isometries:

1. $\hat{a}(K_n) \subseteq M_n$ for all n
2. $\hat{a}(\oplus x_i) = \oplus \hat{a}(x_i)$ for all $x_i \in K_{n_i}$
3. $\hat{a}(\alpha^* x \alpha) = \alpha^* \hat{a}(x) \alpha$ for all $x \in K_n$ and isometries $\alpha \in M_{n,m}$

Definition (DK2019)

Let K be a compact nc convex set. A function $f : K \to \coprod M_n$ is an **nc function** if it is graded, respects direct sums and is equivariant with respect to unitaries:

1. $f(K_n) \subseteq M_n$ for all n
2. $f(\oplus x_i) = \oplus f(x_i)$ for all $x_i \in K_{n_i}$
3. $f(\alpha^* x \alpha) = \alpha^* f(x) \alpha$ for all $x \in K_n$ and unitaries $\alpha \in M_n$

The function f is **affine** if in addition it is equivariant with respect to isometries:

3'. $f(\beta^* x \beta) = \beta^* f(x) \beta$ for all $x \in K_n$ and isometries $\alpha \in M_{n,m}$

Analogous to notion of nc holomorphic function on nc domain defined by Taylor (1973) and Voiculescu (2000).

We write $\mathrm{C}(K)$ for the C*-algebra of continuous nc functions on K, $\mathrm{A}(K)$ for the unital operator system of continuous affine nc functions on K. Elements in $\mathrm{C}(K)$ are "uniform" limits of nc *-polynomials in $\mathrm{A}(K)$.

For $a_1, a_2, a_3 \in A(K)$, define $f \in C(K)$ to be the nc polynomial

$$f = a_1 a_2^2 a_3^* - a_1 a_3^* a_2^2.$$

Then for $x \in K_n$,

$$\begin{aligned} f(x) &= a_1(x)a_2^2(x)a_3(x)^* - a_1(x)a_3(x)^* a_2^2(x) \\ &= x(a_1)x(a_2)x(a_2)x(a_3)^* - x(a_1)x(a_3)^* x(a_2)x(a_2) \in M_n. \end{aligned}$$

Theorem (DK 2019)

We have

$$C(K) = C^*(A(K)) \cong C^*_{max}(A(K)),$$

where $C^*_{max}(A(K))$ is the maximal C-cover of $A(K)$. Moreover, $C(K)^{**}$ is the C*-algebra of bounded nc functions on K.*

Proof uses noncommutative Gelfand representation theorem of Takesaki (1967) and Bichteler (1969).

Categorical (dual) equivalence

Theorem (DK 2019, Webster-Winkler 1999)

A unital operator system with nc state space K is unitally completely order isomorphic to the operator system $\mathrm{A}(K)$. The category of unital operator systems with unital completely positive maps is (dually) equivalent to the category of compact nc convex sets with continuous affine nc maps:

$$\mathrm{A}(K) \quad \longleftrightarrow \quad K$$

More generally, can consider generalized (i.e. potentially non-unital) operator systems of Werner (2002) and Connes-van Suijlekom (2020).

A (generalized) **operator system** is a closed self-adjoint subspace of a C*-algebra (i.e. $S = S^* \subseteq A$).

Definition

The **nc quasistate space** of S is the pair (K, z), where $K = \coprod_{n \le \kappa} K_n$,

$$K_n = \{x : A \to M_n \text{ contractive completely positive}\},$$

and $z \in K_1$ is the zero map.

A pair (K, z) consisting of a compact nc convex set K and a point $z \in K_1$ is **pointed** if (K, z) is the nc quasistate space of the operator system $\mathrm{A}(K, z) \subseteq \mathrm{A}(K)$ of functions that vanish at z.

Theorem (KKM 2021)

An operator system with nc quasistate space (K, z) is isomorphic (completely isometric and completely order isomorphic) to the operator system $\mathrm{A}(K, z)$. The category of operator systems with completely positive maps is (dually) equivalent to the category of pointed compact nc convex sets with continuous pointed affine nc maps:

$$\mathrm{A}(K, z) \quad \longleftrightarrow \quad (K, z)$$

Extreme points

There is a robust notion of extreme point for compact nc convex sets.

Theorem (DK 2019)

A compact nc convex set is the closed nc convex hull of its extreme points.

Extreme points are dual to Arveson's notion of boundary representation of a unital operator system. Proof of existence is difficult (DK 2019, DK 2015, Arveson 2007).

Example

Let K be the **nc d-ball**, i.e. the nc state space of the Cuntz operator system

$$\mathrm{span}\{1, v_1, v_1^* \ldots, v_n, v_n^*\}.$$

The extreme points ∂K are irreducible "cuntz isometries" corresponding to irreducible representations of $\mathcal{O}_d$. Note: $\partial K \subseteq K_{\aleph_0}$.

More generally, if S is a unital operator system with nc state space K and $A = \mathrm{C}^*(S)$, then ∂K can be identified with an (often very complicated) subset of the irreducible representations of A.

Review of some classical Choquet theory

Classical Choquet theory: the study of compact convex sets C via the interplay between $A(C)$ and $\mathrm{C}(C)$.

Let C be a compact convex set. A probability measure $\mu \in \mathrm{P}(C) = S(\mathrm{C}(C))$ **represents** a point $x \in C$ if $\mu|_{A(C)} = \delta_x$. Hence

$$x = \int_C y \, d\mu(y).$$

Theorem (Choquet 1956, Bishop-de Leeuw 1959)

Let C be a compact convex set. For $x \in C$ there is a probability measure μ on C that represents x and is maximal in the **Choquet order***:*

$$\mu \prec \nu \iff \mu(f) \leq \nu(f) \text{ for all convex } f \in C(C)_{sa}.$$

Maximality in Choquet order is an order-theoretic condition equivalent to $\mathrm{supp}(\mu) \subseteq \partial C$ when C is metrizable (and in an appropriate sense more generally).

Definition

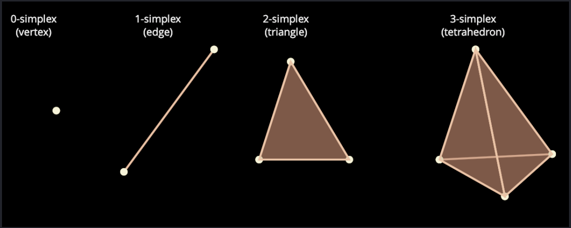
A compact convex set C is a **(Choquet) simplex** if there is a **unique** representing probability measure $\mu_x \in C(C)^*$ for each $x \in K$ that is maximal in the Choquet order.

Intuition: C is a simplex if every point in C can be uniquely expressed as a convex combination of extreme points of C.

For $C \subseteq \mathbb{R}^d$, Caratheodory's theorem implies there are finitely supported Choquet maximal representing measures for $x \in C$,

$$\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i} \iff x = \sum_{i=1}^{n} \alpha_i x_i,$$

and the above statement is literally true.



Application 1: State spaces of unital commutative C*-algebras

A simplex C is a **Bauer simplex** if it has closed extreme boundary ∂C.

Theorem (Bauer 1963)

A function system $A(C)$ is a C-algebra if and only if C is a Bauer simplex.*

The d-simplex is unique up to affine homeomorphism. Hence there is a unique commutative C*-algebra of dimension $d+1$, namely $\mathbb{C}^{d+1}$.

More generally, C is a Bauer simplex if and only if it is affinely homeomorphic to the space of probability measures $P(X)$ on a compact Hausdorff space X, i.e. K is the state space of $C(X)$. So Bauer's theorem implies Gelfand's representation theorem.

Application 2: Dynamical characterization of property (T)

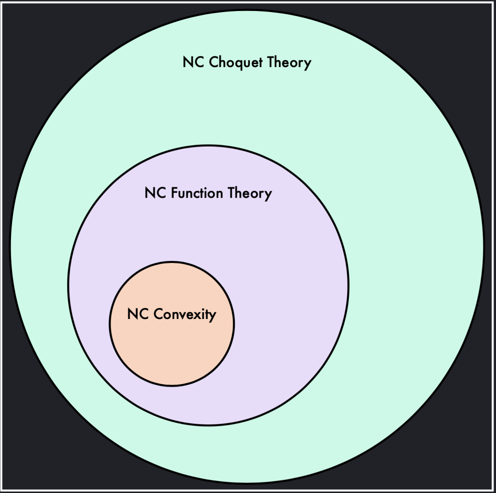
Theorem (Glasner-Weiss 1997)

A group G has property (T) if and only if for every flow (X, G), the set $P(X)^G$ of invariant probability measures is a Bauer simplex.

By Bauer's theorem, equivalent to the statement that G has property (T) if and only if for every commutative C*-dynamical system $(C(X), G)$, the set $P(X)^G$ of invariant states is the state space of (some) commutative C*-algebra.

Noncommutative Choquet theory

Noncommutative Choquet theory: the study of K or $\mathrm{A}(K)$ via the interplay between $A(K)$ and $C(K)$.


NC Choquet Theory
NC Function Theory
NC Convexity

Definition

A self-adjoint nc function $f \in C(K)$ is **convex** if its epigraph

$$\mathrm{Epi}(f) = \coprod_n \{(x, \alpha) : f(x) \leq \alpha\} \subseteq \coprod_n K_n \times M_n$$

is an nc convex set.

Equivalently, $f(\alpha^* x \alpha) \leq \alpha^* f(x) \alpha$ for all $x \in K$ and all isometries α.

Example

Let $I \subseteq \mathbb{R}$ be a compact interval. Define $K = \coprod K_n$ by

$$K_n = \{\alpha \in (M_n)_{sa} : \sigma(\alpha) \subseteq I\}.$$

Then K is a compact nc convex set with $K_1 = I$. A self-adjoint function $f \in C(K)$ is convex as an nc function iff the restriction $f|_{K_1}$ is operator convex, i.e.

$$f(t\alpha + (1-t)\beta) \leq tf(\alpha) + (1-t)f(\beta)$$

for $t \in [0,1]$ and self-adjoint $\alpha, \beta \in M_n$ with $\sigma(\alpha), \sigma(\beta) \subseteq I$.

Essentially the Hansen-Pedersen-Jensen inequality.

An nc state $\mu : C(K) \to M_n$ **represents** a point $x \in K$ if $\mu|_{A(K)} = \delta_x$, i.e.

$$\mu(a) = a(x), \quad \text{for all } a \in \mathrm{A(K)}.$$

Theorem (NC Choquet-Bishop-de Leeuw - DK 2019)

For $x \in K$ there is an nc state $\mu : C(K) \to M_n$ that represents x and is maximal in the **nc Choquet order***:*

$$\mu \prec_{nc} \nu \iff \mu(f) \leq \nu(f) \text{ for all convex } f \in C(K)_{sa}.$$

Theorem (DK 2019)

Maximality in the nc Choquet order implies that $\mathrm{supp}(\mu) \subseteq \partial K$ *in an appropriate sense.*

Theorem (Noncommutative integral representation - DK 2019)

For $x \in K$ there is a nc probability measure λ on K that represents x and is supported on ∂K, meaning that

$$a(x) = \int_K a \, d\lambda, \quad \textit{for all } a \in A(K).$$

Idea: An nc measure is a cp map valued measure. For $f \in C(K)$ and an nc measure λ on K,

$$\int_K f \, d\lambda \approx \sum_{x \in K} \lambda(x)(f(x)).$$

More generally, obtain integral representations of nc states on $C(K)$. Applies to e.g. ucp maps on C*-algebras.

Noncommutative Choquet simplices

Definition

A compact nc convex set K is an **nc (Choquet) simplex** if each point $x \in K$ has a unique nc state $\mu_x : \mathrm{C}(K) \to M_n$ that represents it and is maximal in the nc Choquet order.

Intuition: K is an nc simplex if every point in K can be uniquely expressed as an nc convex combination of extreme points of K.

Theorem (KS 2019)

If C is a (classical) simplex, then there is a unique nc convex set K with $K_1 = C$ and K is an nc simplex.

Hence generalizes classical simplices.

Theorem (KS 2019)

The following are equivalent for a compact nc convex set K:

1. K is an nc Choquet simplex
2. $A(K)^{**}$ is a von Neumann algebra, i.e. $A(K)$ is a **C*-system** in the terminology of Kirchberg-Wassermann
3. $A(K)$ is (c,max)-nuclear in the sense of Kavruk-Paulsen-Todorov-Tomforde, i.e.

$$A(K) \otimes_c S = A(K) \otimes_{max} S$$

for every operator system S.

Generalization of the fact that a compact convex set C is a simplex if and only if $\mathrm{A}(C)^{**}$ is a von Neumann algebra.

Corollary (KS 2019)

Let S be an operator system with nc state space K. If S is a C*-algebra or has the weak expectation property (in particular if it is nuclear) then K is an nc simplex.

Application 1: State spaces of C*-algebras

Definition

A nc Choquet simplex K is an **nc Bauer simplex** if ∂K is closed.

Theorem (KKM 2021, KS 2019)

1. *A compact nc convex set K is affinely homeomorphic to the nc state space of a unital C*-algebra if and only if it is an nc Bauer simplex.*
2. *A pointed compact nc convex set (K, z) is affinely homeomorphic ot the nc quasistate space of a C*-algebra if and only if K is an nc Bauer simplex and $z \in \partial K$.*

Compare to (deep) characterization of state spaces of C*-algebras by Alfsen-Shultz (1978) in terms of compact convex sets with an orientation.

Application 2: Noncommutative dynamical characterization of property (T)

Theorem (KS 2019)

A group G has property (T) if and only if whenever (A, G) is a C-dynamical system with nc state space K (resp. nc quasistate space (K, z)), the set K^G of invariant nc states is an nc Bauer simplex, and hence affinely homeomorphic to the nc state space of (some) C*-algebra.*

Corollary

A group G has property (T) if and only if whenever (A, G) is a C-dynamical system, then the set K_1^G of invariant states (resp. quasistates) is the state space (resp. quasistate space) of a C*-algebra.*

Drury-Arveson space

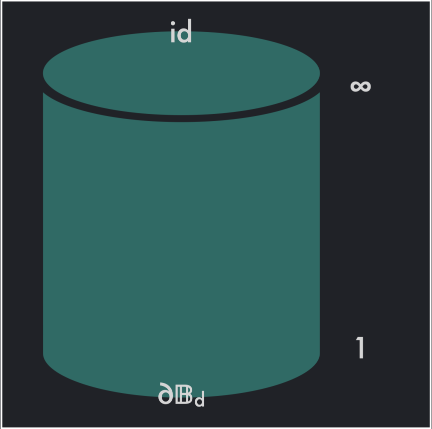
Let H^2_d denote the Drury-Arveson space. Let $M_z = (M_{z_1}, \dots, M_{z_d})$ denote the d-shift,

$$M_{z_i} f = z_i f, \quad f \in H^2_d.$$

Let $K = \sqcup_n K_n$ denote the nc state space of the "Drury-Arveson operator system" $\mathrm{span}\{1, M_{z_1}, M_{z_1}^*, \ldots, M_{z_d}, M_{z_d}^*\}$.

Theorem (Arveson 1998)

The extreme points of K are $\partial K = \{U \,\mathrm{id}\, U^* : U \in U(H_d^2)\} \sqcup \partial\mathbb{B}_d$.



Write $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$. Let $I \leq \mathbb{C}[z]$ be an ideal. The closure $M = \overline{I}$ in H^2_d is a finitely generated $\mathbb{C}[z]$-module via

$$p \cdot f = M_{p(z)} f, \quad f \in M.$$

Let $S = (S_1, \ldots, S_d)$ denote the compression of $M_z = (M_{z_1}, \ldots, M_{z_d})$ to $H^2_d \ominus M$. Let $\mathcal{S} = \text{span}\{1, S_1, S_1^*, \ldots, S_d, S_d^*\}$ and let L denote the nc state space of $\mathcal{S}$.

Theorem (K-Shalit 2015)

The extreme points of L satisfy

$$\partial L \subseteq \{U \, \text{id} \, U^* : U \in U(H^2_d)\} \sqcup \partial V,$$

where $\partial V = \overline{Z(I) \cap \partial \mathbb{B}_d}$.

Question

Is this an equality?

Motivation

Theorem (K-Shalit 2015)

The $\mathbb{C}[z]$-module M is essentially normal if and only if the operator system $\mathcal{S} = \mathrm{span}\{1, S_1, S_1^*, \dots, S_d, S_d^*\}$ is hyperrigid.

Equivalent to the statement that the restriction $\pi|_{\mathcal{S}}$ of every representation π of $\mathrm{C}^*(\mathcal{S})$ has a unique extension to a ucp map (which is necessarily π).

Hyperrigidity implies equality:

$$\partial L = \{U \,\mathrm{id}\, U^* : U \in U(H^2_d)\} \sqcup \partial V,$$

Hence a negative anwer would provide a counterexample to the Arveson-Douglas essential normality conjecture.

On the other hand, a positive answer when essential normality fails would provide a counterexample to Arveson's hyperrigidity conjecture

Thanks!