Quotients of the Drury-Arveson space and their classification in terms of complex geometry

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Based on joint works with Davidson, Hartz, Kerr, McCarthy, Ofek, Pandey, Ramsey and Salomon.

Other contributors: Alpay, Arcozzi, Hartz, Lupini, Putinar, Rochberg, Sawyer, Vinnikov...

Related works in the setting of **noncommutative function theory**: Kakariadis, Muhly, Popescu, Salomon, Shamovich, Solel...

Recommended reading:
1) G. Salomon and O.M. Shalit, *The Isomorphism Problem for Complete Pick Algebras: A Survey*, OTAA 2016.

2) D, Ofek, S. Pandey, O.M. Shalit, *Distance between reproducing kernel Hilbert spaces and geometry of finite sets in the unit ball*, JMAA 2021.

A corollary of Gelfand's theory

Theorem (1940s)

Let X and Y be compact Hausdorff topological spaces.
The algebra $C(X)$ is isomorphic to $C(Y)$ if and only if $X \cong Y$.
In fact, $\varphi : C(X) \to C(Y)$ is an isomorphism, then there exists a homeomorphism $\alpha : Y \to X$ such that

$$\varphi(f) = f \circ \alpha \ , \ f \in C(X)$$

Proof

Given $\varphi : C(X) \to C(Y)$.

$\varphi^* : C(Y)^* \to C(X)^*$ by $\varphi^*(\rho) = \rho \circ \varphi$.

Key: $X \cong$ the maximal ideal space $\mathfrak{M}(C(X))$ of $C(X)$

$$X \ni x \leftrightarrow \rho_x \in \mathfrak{M}(C(X))$$

$$\rho_x : f \mapsto f(x)$$

φ^* maps points of Y to points of X. $\alpha := \varphi^*\big|_Y$ is a homeomorphism.

Now compute:

$$\varphi(f)(y) = \rho_y(\varphi(f)) = \varphi^*(\rho_y)(f) = f(\alpha(y))$$

Thus $\varphi(f) = f \circ \alpha$. (The converse is obvious.)

Our interest: classification results in this spirit for other algebras — quotients of the multiplier algebra of Drury-Arveson space.

Drury-Arveson space and its quotients

The Drury-Arveson space H^2_d is the Hilbert function space on $\mathbb{B}_d$ with kernel $k(z,w) = k_w(z) = (1 - \langle z, w \rangle)^{-1}$. The multiplier algebra is

$$\mathcal{M}_d := \mathrm{Mult}(H^2_d) = \{f : \mathbb{B}_d \to \mathbb{C} : fh \in H^2_d \text{ for all } h \in H^2_d\}$$

Quotients of H^2_d

A variety in $\mathbb{B}_d$ is the zero set of multipliers, i.e. a set of the form

$$V = \{z \in \mathbb{B}_d : f(z) = 0 \text{ for all } f \in F\} \qquad (\text{for some } F \subseteq \mathcal{M}_d)$$

A quotient of H^2_d is a subspace of the form

$$\mathcal{H}_V = \overline{\mathrm{span}}\{k_\lambda : \lambda \in V\} = H^2_d \ominus K_V \cong H^2_d / K_V$$

where $K_V = \{h \in H^2_d : h(z) = 0 \text{ for all } z \in V\}$.

Quotients of $\mathrm{Mult}(H^2_d)$

Quotients of $\mathcal{M}_d$

A quotient of $\mathcal{M}_d$ is an algebra of the form

$$\mathcal{M}_V = \mathcal{M}_d\big|_V := \{f\big|_V : f \in \mathcal{M}_d\}$$

Fact: $\mathcal{M}_V$ is an operator algebra and

$$\mathcal{M}_V = \mathrm{Mult}(\mathcal{H}_V) \cong \mathcal{M}_d / J_V$$

where $J_V = \{f \in \mathcal{M}_d : f(z) = 0 \text{ for all } z \in V\}$.

Theorem (Agler-McCarthy 2000)

Every irreducible complete Pick space is (up to rescaling) of the form $\mathcal{H}_V$ for some (multiplier) variety $V \subset \mathbb{B}_d$ and for some $d \in \mathbb{N} \cup \{\infty\}$.
Every multiplier algebra of an irreducible Pick space is of the form $\mathcal{M}_V$.

The isomorphism problem for $\mathcal{M}_V$

We are interested in the analogue of Gelfand's theorem for the quotient spaces $\mathcal{H}_V$ and quotient algebras $\mathcal{M}_V = \mathrm{Mult}(H^2_d)\big|_V = \mathrm{Mult}(\mathcal{H}_V)$.

Problem A

Let $V \subset \mathbb{B}_d$ be a variety. How does the **geometry** of V determine the **structure** of $\mathcal{M}_V$?

More concretely:

Problem B

Let $V, W \subset \mathbb{B}_d$ be two varieties. When are $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic? Isometrically isomorphic? Completely Isometrically isomorphic? Similar/unitarily equivalent?

(Note: these algebras are semisimple, so algebraic isomorphisms are automatically bounded.)

The maximal ideal space of $\mathcal{M}_V$

$\mathfrak{M}(\mathcal{M}_V)$ is usually **huge** (think about $H^\infty = \mathcal{M}_{\mathbb{D}}$).
Let

$$\pi : \mathfrak{M}(\mathcal{M}_V) \to \overline{\mathbb{B}}_d$$

$$\pi(\rho) = (\rho(z_1), \rho(z_2), \ldots, \rho(z_d)) \in \overline{\mathbb{B}}_d$$

For every $\lambda \in V$, there exists $\rho_\lambda \in \pi^{-1}(\lambda)$ given by

$$\rho_\lambda(f) = f(\lambda)$$

Note:

$$\pi(\rho_\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_d) = \lambda$$

We identify $V \subset \mathfrak{M}(\mathcal{M}_V)$ — that's what we can see.

The maximal ideal space of $\mathcal{M}_V$

Theorem (Davidson-Pitts 1998, Davdison-Ramsey-S)

There is a continuous projection $\pi : \mathfrak{M}(\mathcal{M}_V) \to \overline{\mathbb{B}_d}$ *given by*

$$\pi(\rho) = (\rho(z_1), \rho(z_2), \dots, \rho(z_d)) \ , \ \ \rho \in \mathfrak{M}(\mathcal{M}_V).$$

For $\lambda \in V$ *there is a unique weak-$*$ continuous* $\rho_\lambda \in \pi^{-1}(\lambda)$ *given by* $\rho_\lambda(f) = f(\lambda)$. *Every weak-$*$ continuous character arises this way.*
If $d < \infty$, *then*

$$\pi(\mathfrak{M}(\mathcal{M}_V)) \cap \mathbb{B}_d = V, \tag{1}$$

$$\pi^{-1}(\lambda) = \{\rho_\lambda\}, \text{ for all } \lambda \in V. \tag{2}$$

If $d = \infty$, then (1) or (2) may fail.

Isometric isomorphism = “conformal" equivalence

Theorem (Davidson-Ramsey-S 2015)

Let $V, W \subset \mathbb{B}_d$ *be varieties,* $d < \infty$. *The following assertions are equivalent:*

(i) $\mathcal{M}_V$ *and* $\mathcal{M}_W$ *are unitarily equivalent.*

(ii) $\mathcal{M}_V$ *and* $\mathcal{M}_W$ *are (completely) isometrically isomorphic.*

(iii) *There exists a biholomorphic automorphism* $\alpha \in \mathrm{Aut}(\mathbb{B}_d)$ *such that*

$$\alpha(W) = V$$

Proof:

(i) $\Rightarrow$ (ii) is immediate.

Proof (cont'd):

(ii) $\Rightarrow$ (iii): **IF** φ^* maps point evaluation at λ to point evaluation at μ, then

$$\varphi(z_i)(\lambda) = \rho_\lambda \circ \varphi(z_i) = \varphi^*(\rho_\lambda)(z_i) = \rho_\mu(z_i) = \mu_i$$

Thus $\mu = (\varphi(z_1)(\lambda), \dots, \varphi(z_d)(\lambda)) =: \alpha(\lambda)$ where $\alpha = (\varphi(z_1), \dots, \varphi(z_d))$. α is analytic on $\mathbb{B}_d$, and $\alpha(W) \subseteq V$. Using complete contractivity + invertibility + SCV one gets $\alpha \in \mathrm{Aut}(\mathbb{B}_d)$ and $\alpha(W) = V$.

We show φ^* preserves point evaluations using a mixture of operator algebras and SCV techniques.

(iii) $\Rightarrow$ (i): We write down a unitary $U : \mathcal{H}_W \to \mathcal{H}_V$

$$U k_\lambda = c_\lambda k_{\alpha(\lambda)}$$

and this unitary implements an isomorphism between $\mathcal{M}_V$ and $\mathcal{M}_W$.

Note that the proof passes through $\mathcal{H}_V \cong \mathcal{H}_W$.

Algebraic isomorphism for homogeneous varieties

For homogeneous varieties* in $d < \infty$

$$\text{algebraic isomorphism} \;=\; \text{biholomorphic equivalence}$$

Theorem (Davidson-Ramsey-S 2011, Hartz 2012)

Let $V, W \subset \mathbb{B}_d$ be homogeneous varieties with $d < \infty$. Then the following are equivalent:

(i) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are similar ($\exists T \in B(\mathcal{H}_V, \mathcal{H}_W). \mathcal{M}_V = T^{-1}\mathcal{M}_W T$).*

(ii) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic.*

(iii) *V and W are biholomorphically equivalent.*

(iv) *There is an invertible linear map on $\mathbb{C}^d$ which maps V onto W.*

* - A homogeneous variety is the common vanishing locus of homogeneous polynomials e.g. $p(z) = z_1 z_2 + z_3^2$.

Theorem (Davidson-Ramsey-S 2011, Hartz 2012)

Let $V, W \subset \mathbb{B}_d$ be homogeneous varieties with $d < \infty$. Then the following are equivalent:

(i) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are similar ($\exists T \in B(\mathcal{H}_V, \mathcal{H}_W).\mathcal{M}_V = T^{-1}\mathcal{M}_W T$).*

(ii) *$\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic.*

(iii) *V and W are biholomorphically equivalent.*

(iv) *There is an invertible linear map on $\mathbb{C}^d$ which maps V onto W.*

Proof: (i) $\Rightarrow$ (ii) is immediate.
(ii) $\Rightarrow$ (iii): As in isometric case, we show that φ^* preserves weak-$*$ continuous characters (but different argument).
(iii) $\Rightarrow$ (iv): Use elementary **several complex variables**.
(iv) $\Rightarrow$ (i): If $A : V \to W$, define $T : k_\lambda \mapsto k_{A\lambda}$.
One shows that T extends to a bounded isomorphism $\mathcal{H}_V \to \mathcal{H}_W$ (this is difficult! Call Michael Hartz...).

Note that here too the proof passes through $\mathcal{H}_V \cong_w \mathcal{H}_W$.

Algebraic isomorphism $\Rightarrow$ biholomorphic equivalence

Theorem (Davidson-Ramsey-S 2015)

Let $V, W \subset \mathbb{B}_d$ be varieties in $\mathbb{B}_d$ with $d < \infty$ which are the union of finitely many irreducible varieties and a discrete variety. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic, then V and W are biholomorphic.

Questions

1. **What about the converse?**
2. Are the technical conditions (finite union of irreducibles, $d < \infty$) really needed?

A positive result for discs

Theorem (Alpay-Putinar-Vinnikov 2003)

Let $\alpha : \mathbb{D} \to V \subset \mathbb{B}_d$ be a biholomorphism, $d < \infty$. If

(1) *α extends to an injective C^2 function on $\overline{\mathbb{D}}$,*

(2) *$\alpha'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$,*

(3) *$||\alpha(z)|| = 1$ if and only if $|z| = 1$, and*

Then $\mathcal{M}_V$ is isomorphic to $H^\infty(\mathbb{D})$. In fact $\mathcal{M}_V = H^\infty(V)$.

Extended to planar domains (in place of $\mathbb{D}$) by Arcozzi-Rochberg-Sawyer (2008), and to finite (open) Riemann surfaces by Kerr-McCarthy-S (2013).

An application

Let $\alpha : \mathbb{D} \to V \subset \mathbb{B}_d$ be a biholomorphism, $d < \infty$. By [APV03]

$$H^\infty(V) = \mathcal{M}_V := \mathrm{Mult}(H^2_d)\big|_V$$

Corollary (Alpay-Putinar-Vinnikov 2003)

Let $\alpha : \mathbb{D} \to V \subset \mathbb{B}_d$ be a biholomorphism as in the theorem. Then for every $f \in H^\infty(V)$ there exists $\tilde{f} \in \mathrm{Mult}(H^2_d)$ such that $\tilde{f}\big|_V = f$.

One also has control on the multiplier norm of $\tilde{f}$.
Since $\mathrm{Mult}(H^2_d) \subsetneq H^\infty(\mathbb{B}_d)$, this is sharpening of Henkin's difficult and deep extension theorem in the one dimensional case.

Kerr-McCarthy-S generalized this to "nice" one-dimensional varieties in the unit ball.

A negative result for discs

Theorem (Davidson-Hartz-S 2015)

There exists a rational function α mapping $\mathbb{D}$ onto $V \subset \mathbb{B}_2$ with poles outside $\overline{\mathbb{D}}$ as in the Alpay-Putinar-Vinnikov theorem, except for $\alpha(1) = \alpha(-1)$.
In this case, $\mathcal{M}_V$ is not isomorphic to H^∞, and $\alpha^{-1} \notin \mathcal{M}_V$.

So the answer to the question whether $\mathcal{M}_V \simeq H^\infty$ for V an analytic disc is **no**, even when $d < \infty$. In other words

$$\text{biholomorphism} \not\Rightarrow \text{isomorphism}$$

To understand what's really going on we need some more definitions...

Automorphisms of $\mathbb{B}_d$ and the pseudohyperbolic distance

For $w \in \mathbb{B}_d$

$$\Psi_w(z) = \frac{w - P_w z - (1 - \|w\|^2)^{1/2} P_w^{\perp} z}{1 - \langle z, w \rangle}$$

is a holomorphic automorphism of the ball that maps w to 0 and 0 to w. Here P_w is the projection onto the span of w and $P_w^{\perp} = I - P_w$.

The pseudohyperbolic metric d_{ph} on the open unit ball $\mathbb{B}_d$ is given by

$$d_{\mathrm{ph}}(z, w) := \|\Psi_w(z)\| = \|\Psi_z(w)\|, \qquad z, w \in \mathbb{B}_d$$

In particular, when $d = 1$, we have

$$\Psi_w(z) = \frac{w - z}{1 - \overline{w} z} \text{ and } d_{\mathrm{ph}}(z, w) = \left| \frac{w - z}{1 - \overline{w} z} \right|$$

A negative result for discs (revisited)

Theorem (Davidson-Hartz-S 2015)

There exists a rational function α mapping $\mathbb{D}$ onto $V \subset \mathbb{B}_2$ with poles outside $\overline{\mathbb{D}}$ as in the Alpay-Putinar-Vinnikov theorem, except for $\alpha(1) = \alpha(-1)$.
In this case, $\mathcal{M}_V$ is not isomorphic to H^∞, and $\alpha^{-1} \notin \mathcal{M}_V$.

Proof: We show that if $\varphi : \mathcal{M}_V \to \mathcal{M}_W$ is an isomorphism, then $\alpha = \varphi^*$ is bi-Lipschitz w.r.t. pseudo-hyperbolic distance, that is

$$C^{-1} d_{\mathrm{ph}}(z, w) \leq d_{\mathrm{ph}}(\alpha(z), \alpha(w)) \leq C d_{\mathrm{ph}}(z, w)$$

We show that such a map α is not bi-Lipschitz.

A stronger notion of biholomorphism

Definition

A multiplier biholomorphism is a biholomorphism $\alpha : V \to W$ such that the coordinates of α are in $\mathcal{M}_V$, and the coordinates of α^{-1} are in $\mathcal{M}_W$.

Theorem (Davidson-Ramsey-S 2015, Davidson-Hartz-S 2015)

Let $V, W \subset \mathbb{B}_d$ be varieties in $\mathbb{B}_d$ with $d < \infty$ which are the union of finitely many irreducible varieties and a discrete variety. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are isomorphic, then V and W are multiplier biholomorphic and bi-Lipschitz w.r.t. pseudo-hyperbolic distance.

Question

If V and W are multiplier biholomorphic, is $\mathcal{M}_V$ isomorphic to $\mathcal{M}_W$?

..it depends...

More recent stuff...

Noncommutative versions

Recall (from Mike Jury's series):
$\mathcal{H}^2_{nc}$ is a noncommutative RKHS with multiplier algebra

$$\mathrm{Mult}(\mathcal{H}^2_{nc}) = \mathcal{H}^\infty_{nc}(\text{row ball})$$

Can consider $\mathcal{H}^2_{nc}\big|_V$ and $\mathrm{Mult}(\mathcal{H}^2_{nc})\big|_V$ where V an nc variety.

When is $\mathrm{Mult}(\mathcal{H}^2_{nc})\big|_V$ (isometrically) isomorphic to $\mathrm{Mult}(\mathcal{H}^2_{nc})\big|_W$?

1) G. Salomon, O.M. Shalit and E. Shamovich, *Algebras of bounded noncommutative analytic functions on subvarieties of the noncommutative unit ball*, TAMS 2018.

2) G. Salomon, O.M. Shalit and E. Shamovich, *Algebras of noncommutative functions on subvarieties of the noncommutative ball: The bounded and completely bounded isomorphism problem*, JFA 2020.

Apologies: two different papers with same authors and similar long titles, confusing!

The new stuff...

A quantitative version of the problem

We now switch to varieties $V, W \subseteq \mathbb{B}_d$ that are finite sets of points. In this situation we have a complete answer to the question.

Theorem (DRS 2015, Rochberg 2019)

Let V, W be two finite subsets of $\mathbb{B}_d$.

(i) *$\mathcal{M}_V$ is completely isometrically isomorphic to $\mathcal{M}_W$ if and only if there is an automorphism $\Phi \in \mathrm{Aut}(\mathbb{B}_d)$ such that $\Phi(V) = W$.*

(ii) *$\mathcal{M}_V$ is isomorphic to $\mathcal{M}_W$ if and only if there is a biholomorphism from V onto W (and this happens if and only if $|V| = |W|$).*

Question: what happens if we move the points in (i) just a little a bit?

Distances between subsets and algebras

Recall that the Hausdorff distance between two subsets E, F in a metric space (X, d) is defined to be

$$d_H(E,F) = \max\{\max_{x \in E} d(x,F), \max_{y \in F} d(y,E)\}$$

Definition (The conformal invariant Hausdorff distance)

For $V, W \subseteq \mathbb{B}_d$ we define

$$\tilde{d}_H(V,W) = \inf\{d_H(V, \Phi(W)) : \Phi \in \mathrm{Aut}(\mathbb{B}_d)\}$$

Definition (The “Banach Mazur" multiplier distance)

For two multiplier algebras $\mathcal{A}, \mathcal{B}$ we define

$$\rho_M(\mathcal{A},\mathcal{B}) = \log \inf\{\|\varphi\|_{cb}\|\varphi^{-1}\|_{cb} : \varphi : \mathcal{A} \to \mathcal{B} \text{ isomorphism* }\}$$

* We mean multiplier algebra isomorphism, i.e. $\varphi(f) = f \circ F$ for some F.

Quantitative version of the isomorphism problem

Theorem (Ofek-Pandey-S 2021)

Let $V \subseteq \mathbb{B}_d$ be a finite set with $|V| = n$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $W \subseteq \mathbb{B}_d$ and $|W| = n$ then

(i) $\tilde{d}_H(V, W) < \delta \Longrightarrow \rho_M(\mathcal{M}_V, \mathcal{M}_W) < \varepsilon$.

(ii) $\rho_M(\mathcal{M}_V, \mathcal{M}_W) < \delta \Longrightarrow \tilde{d}_H(V, W) < \varepsilon$.

Proof: The proofs in both directions go through the Hilbert spaces.

For RKHSs $\mathcal{H}_1, \mathcal{H}_2$ on sets X_1, X_2 we define

$$\rho_{RK}(\mathcal{H}_1, \mathcal{H}_1) = \log \inf\{\|T\|\|T^{-1}\| : T : \mathcal{H}_1 \to \mathcal{H}_2 \text{ isomorphism}^* \}$$

* we mean isomorphism of reproducing kernel Hilbert spaces, i.e. T is bounded and is determined by

$$Tk_x^1 = c_\lambda k_{F(x)}^2 \quad , \quad x \in X_1$$

where $F : X_1 \to X_2$ is a bijection.

Quantitative version of the isomorphism problem II

Theorem (Ofek-Pandey-S 2021)

Let $V \subseteq \mathbb{B}_d$ be a finite set with $|V| = n$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $W \subseteq \mathbb{B}_d$ and $|W| = n$ then

(i) $\tilde{d}_H(V,W) < \delta \Longrightarrow \rho_{RK}(\mathcal{H}_V, \mathcal{H}_W) < \varepsilon$.

(ii) $\rho_{RK}(\mathcal{H}_V, \mathcal{H}_W) < \delta \Longrightarrow \tilde{d}_H(V,W) < \varepsilon$.

(iii) $\rho_{RK}(\mathcal{H}_V, \mathcal{H}_W) < \delta \Longrightarrow \rho_M(\mathcal{M}_V, \mathcal{M}_W) < \varepsilon$.

(iv) $\rho_M(\mathcal{M}_V, \mathcal{M}_W) < \delta \Longrightarrow \rho_{RK}(\mathcal{H}_V, \mathcal{H}_W) < \varepsilon$.

Proof: (i) is basic. (ii) requires some nice idea. Note: **it does NOT hold for all** reasonable RKHSs of analytic functions, it depends on the kernel in some way (Ofek-Sofer, "Three classification results...", CAOT 2021).
(iii) is standard: $T : \mathcal{H}_V \to \mathcal{H}_W$, $T(k_v) = c_v k_{F(v)}$ gives $\varphi : \mathcal{M}_V \to \mathcal{M}_W$

$$\varphi : M_f \mapsto (T^{-1})^* M_f T^* = M_{f \circ F}$$

and $\|\varphi\|_{cb} \leq \|T\| \|T^{-1}\|$.
(iv) is the deepest part, depends CNP (think H^∞, H^2, L^2_a).

Thank you!