OPERATORS ON THE DRURY-ARVESON SPACE

Jingbo Xia

Department of Mathematics
SUNY Buffalo

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The story of Drury-Arveson space goes back to the von Neumann inequality for contractions.

A linear operator T on a Hilbert space $\mathcal{H}$ is called a contraction if $\|T\| \leq 1$, i.e., if $\|Tx\| \leq \|x\|$ for every vector $x \in \mathcal{H}$.

von Neumann's famous inequality (1951): If T is a contraction, then

$$\|p(T)\| \leq \sup_{|z|\leq 1} |p(z)|$$

for every polynomial p.

Generalization to operator tuples:

Suppose that $(T_1, \ldots, T_n)$ is a commuting tuple of operators, and suppose that each T_i is a contraction, $i = 1, \ldots, n$. Then one might expect that the inequality

$$(*) \qquad \|p(T_1, \ldots, T_n)\| \leq \sup_{|z_1| \leq 1, \ldots, |z_n| \leq 1} |p(z_1, \ldots, z_n)|$$

holds for every n-variable polynomial p.

This is true in the case $n = 2$ (Ando, 1963) . But this is false for $n \geq 3$ (Varopoulos, 1973).

This raises the following issues:

• What is the right notion of contraction for a commuting tuple of operators?

• What is the right domain on which to consider the problem?

• What is the right norm for the "right-hand side"? (Note that the right-hand side of $(*)$ is $\|p\|_\infty$, the supremum norm of p on the domain in question. Varopoulos' construction shows that for general n, $\|p\|_\infty$ is too small.)

All of these were figured out by 1978.

A commuting tuple $(A_1, \ldots, A_n)$ on a Hilberts space $\mathcal{H}$ is said to be a row contraction if

$$A_1 A_1^* + \cdots + A_n A_n^*$$

is a contraction. Equivalently, $(A_1, \ldots, A_n)$ is a row contraction if

$$\|A_1 x_1 + \cdots + A_n x_n\|^2 \leq \|x_1\|^2 + \cdots + \|x_n\|^2$$

for all $x_1, \ldots, x_n \in \mathcal{H}$. Note that this is stronger than requiring that each individual A_i be a contraction, in fact much stronger.

von Neumann inequality for row contractions:

Drury (1978), Arveson (1998) : If $(A_1, \ldots, A_n)$ is a commuting row contraction, then

$$\|p(A_1, \ldots, A_n)\| \leq \|p\|_{\mathcal{M}}$$

for every $p \in \mathbf{C}[z_1, \ldots, z_n]$.

The right-hand side, $\|p\|_{\mathcal{M}}$, has to be explained on a new space, the Drury-Arveson space H^2_n.

The space H^2_n consists of analytic functions on $\mathbf{B}$. The inner product on H^2_n is defined by the formula

$$\langle h, g\rangle = \sum_{\alpha\in\mathbf{Z}^n_+} \frac{\alpha!}{|\alpha|!} a_\alpha \bar{b}_\alpha$$

for

$$h(\zeta) = \sum_{\alpha\in\mathbf{Z}^n_+} a_\alpha \zeta^\alpha \quad \text{and} \quad g(\zeta) = \sum_{\alpha\in\mathbf{Z}^n_+} b_\alpha \zeta^\alpha,$$

where we use the standard multi-index notation. For $z \in \mathbf{B}$, define

$$K_z(\zeta) = \frac{1}{1 - \langle\zeta, z\rangle}.$$

K_z is the reproducing kernel for the Drury-Arveson space H^2_n. That is, if $h \in H^2_n$ and $z \in \mathbf{B}$, then $h(z) = \langle h, K_z\rangle$.

This space H^2_n has quite a bit of history.

In 1977, Lubin used this space to produce the first example of a tuple of commuting subnormal operators that does not admit a joint normal extension. At present, this is the earliest known appearance of the space H^2_n.

The space H^2_n was then introduced by Drury in 1978 and by Arveson in 1998 for the purpose of generalizing von Nuemann's original inequality to the multi-variable setting.

After Arveson's 1998 paper, H^2_n has been the subject of intense studies.

One of Arveson's main contributions in his 1998 paper is the introduction of multipliers for H_n^2. An $f \in H_n^2$ is said to be a multiplier of H_n^2 if

$$fg \in H_n^2 \quad \text{for every} \quad g \in H_n^2.$$

We will write $\mathcal{M}$ for the collection of multipliers of H_n^2.

For every $f \in \mathcal{M}$, the multiplication operator M_f is bounded on H_n^2 by the closed graph theorem.

Obviously, we have $\mathbf{C}[z_1, \ldots, z_n] \subset \mathcal{M}$.

For each $f \in \mathcal{M}$, define

$$\|f\|_{\mathcal{M}} = \|M_f\| = \sup\{\|fg\| : g \in H_n^2, \|g\| \leq 1\}.$$

This is called the multiplier norm of f.

Recall von Neumann inequality for commuting row contractions:

$$(**) \qquad \|p(A_1, \ldots, A_n)\| \leq \|p\|_{\mathcal{M}}.$$

It is the multiplier norm of $p \in \mathbf{C}[z_1, \ldots, z_n]$ that appears on the right-hand side.

If $n = 1$, then $(**)$ reverts back to von Neumann's original inequality for a single contraction. This corresponds to the fact that H_1^2 is the familiar Hardy space H^2 of analytic functions on the unit disc D. Moreover, if $n = 1$, then $\|p\|_{\mathcal{M}} = \|p\|_\infty$.

Recall that the inner product on one-variable Hardy space H^2 is given by the formula

$$\langle h, g \rangle = \int h(\tau)\overline{g(\tau)}dm(\tau),$$

where dm is the Lebesgue measure on the unit circle $\mathbf{T} = \{\tau \in \mathbf{C} : |\tau| = 1\}$.

Alternately, one can describe H^2 as the norm closure of the collection of polynomials in $L^2(\mathbf{T}, dm)$.

The point is this: There is an L^2 naturally associated with H_1^2.

Suppose that $n \geq 2$.

Then there is no constant $0 < C < \infty$ such that $\|p\|_{\mathcal{M}} \leq C\|p\|_\infty$ for every $p \in \mathbf{C}[z_1, \ldots, z_n]$. Here $\|p\|_\infty = \sup_{z \in \mathbf{B}} |p(z)|$. That is, the multiplier norm $\|\cdot\|_{\mathcal{M}}$ is strictly stronger than $\|\cdot\|_\infty$. Both Drury and Arveson knew this in their original papers.

Arveson (1998): There is no L^2 naturally associated with H_n^2. More specifically, the tuple

$$(M_{\zeta_1}, \cdots, M_{\zeta_n})$$

of multiplication by the coordinate functions on H_n^2 is not jointly subnormal.

This means that H_n^2 is fundamentally different from the more familiar reproducing-kernel Hilbert spaces such as the Hardy space and the Bergman space.

Recall that a commuting tuple $(S_1, \cdots, S_n)$ of operators on a Hilbert space $\mathcal{H}$ is said to be jointly subnormal if there is a Hilbert space $\mathcal{L}$ containing $\mathcal{H}$ and a commuting tuple of normal operators $(N_1, \cdots, N_n)$ on $\mathcal{L}$ such that $\mathcal{H}$ is invariant under each N_i and such that each S_i is the restriction of N_i to $\mathcal{H}$; that is,

$$S_i x = N_i x \quad \text{for every} \quad x \in \mathcal{H},$$

$i = 1, \ldots, n$.

Drury's proof: In order to prove

$$(**) \qquad \|p(A_1,\dots,A_n)\| \le \|p\|_{\mathcal{M}},$$

it suffices to consider commuting tuples $(A_1,\dots,A_n)$ for which that is an $r \in [0,1)$ such that

$$\langle (A_1A_1^* + \cdots + A_nA_n^*)x, x\rangle \le r^2\|x\|^2, \quad x \in \mathcal{H}.$$

For such a tuple, we can resolve the identity operator 1 in the form

$$\sum_{\alpha\in\mathbf{Z}_+^n} \frac{|\alpha|!}{\alpha!} A^\alpha (1 - A_1A_1^* - \cdots - A_nA_n^*) A^{*\alpha} = 1.$$

This enables us to define an isometry $Z : \mathcal{H} \to H_n^2 \otimes \mathcal{H}$:

$$(Zx)(\zeta) = \sum_{\alpha\in\mathbf{Z}_+^n} \frac{|\alpha|!}{\alpha!} (1 - A_1A_1^* - \cdots - A_nA_n^*)^{1/2} A^{*\alpha} x \zeta^\alpha.$$

It is then straightforward to verify that

$$Zp(A_1^*, \ldots, A_n^*) = (p(M_{\zeta_1}^*, \ldots, M_{\zeta_n}^*) \otimes 1)Z$$

for every $p \in \mathbf{C}[z_1, \ldots, z_n]$. This implies $(**)$.

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The Drury-Arveson space H_n^2 is just one member in a family of reproducing-kernel Hilbert spaces of analytic functions on the unit ball $\mathbf{B}$.

For each real number $-n \leq t < \infty$, we can define $\mathcal{H}^{(t)}$, the Hilbert space of analytic functions on $\mathbf{B}$ with the reproducing kernel

$$K_z^{(t)}(\zeta) = \frac{1}{(1 - \langle \zeta, z \rangle)^{n+1+t}}, \quad z, \zeta \in \mathbf{B}.$$

Alternately, one can describe $\mathcal{H}^{(t)}$ as the completion of $\mathbf{C}[z_1, \ldots, z_n]$ with respect to the norm $\| \cdot \|_t$ arising from the inner product $\langle \cdot, \cdot \rangle_t$ defined according to the following rules:
$\langle \zeta^\alpha, \zeta^\beta \rangle_t = 0$ whenever $\alpha \neq \beta$,

$$\langle \zeta^\alpha, \zeta^\alpha \rangle_t = \frac{\alpha!}{\prod_{j=1}^{|\alpha|}(n + t + j)}$$

if $\alpha \in \mathbf{Z}_+^n \backslash \{0\}$, and $\langle 1, 1 \rangle_t = 1$.

One can think of the parameter t as the "weight" of the space, although t can be negative.

We have

$$\begin{aligned} \mathcal{H}^{(0)} &= L_a^2(\mathbf{B}, dv), && \text{the Bergman space,} \\ \mathcal{H}^{(-1)} &= H^2(S), && \text{the Hardy space on the sphere,} \\ \mathcal{H}^{(-n)} &= H_n^2, && \text{the Drury-Arveson space.} \end{aligned}$$

In fact, for each $-1 < t < \infty$, $\mathcal{H}^{(t)}$ is a weighted Bergman space.

One can think of the Bergman space $\mathcal{H}^{(0)}$ as a benchmark, against which the other spaces in the family should be compared.

For all $-n \le s \le t < \infty$, we have the inclusion $\mathcal{H}^{(s)} \subset \mathcal{H}^{(t)}$.

Arazy and G. Zhang (2003): Similar to the case of Drury-Arveson space, if $-n < t < -1$, then the tuple of multiplication operators $(M_{\zeta_1}, \ldots, M_{\zeta_n})$ on $\mathcal{H}^{(t)}$ is not jointly subnormal.

In other words, if $-n < t < -1$, then $\mathcal{H}^{(t)}$ is more like the Drury-Arveson space t $\mathcal{H}^{(-n)}$ than the Hardy space $\mathcal{H}^{(-1)}$.

The practical consequence of this is that it is difficult to do estimates on $\mathcal{H}^{(t)}$ if $-n \leq t < -1$, since there is no measure or L^2 associated with these spaces.

Consequently, results on $\mathcal{H}^{(t)}$, $-n \leq t < -1$, are hard to obtain.

Thus, while we know quite a about these spaces by now, there is plenty more that we don't know. In fact, there are some seemingly elementary questions about these spaces to which we do not know the answer.

The inner-function problem.

Arveson was the first to notice that, when $n \geq 2$, H_n^2 does not contain $H^\infty(\mathbf{B})$, the collection of bounded analytic function on $\mathbf{B}$. He explicitly constructed an $f \in H^\infty(\mathbf{B})$ that does not belong to H_n^2. This construction was based on the function

$$\theta(\zeta_1, \ldots, \zeta_n) = \zeta_1 \cdots \zeta_n$$

on $\mathbf{B}$. Arveson observed that

$$\|\theta^k\|_\infty = \frac{1}{n^{kn/2}} \quad \text{while} \quad \|\theta^k\|_{H_n^2} = \left(\frac{(k!)^n}{(nk)!}\right)^{1/2} \approx \frac{k^{(n-1)/4}}{n^{kn/2}}.$$

Once this is seen, for $n \geq 2$ it is easy to come up with coefficients $a_0, a_1, \ldots a_k, \ldots$ such that $f = \sum_{k=0}^\infty a_k \theta^k$ is in $H^\infty(\mathbf{B})$ but not in H_n^2. In fact, one can even require f to be continuous on the closure of the unit ball $\mathbf{B}$.

It should be mentioned that examples of $f \in H^\infty(\mathbf{B})$, $f \notin H^2_n$ actually existed in plain sight. From the last chapter of Rudin's famous book "Function theory in the unit ball of $\mathbf{C}^n$" we know that when $n \geq 2$, if u is a non-constant inner function on $\mathbf{B}$, then $|\nabla u|$ is not square-integrable with respect to the volume measure on $\mathbf{B}$. Using the spaces introduced in this section, we can rephrase this result as saying that if u is a non-constant inner function on $\mathbf{B}$, then $u \notin \mathcal{H}^{(-2)}$. In particular, $\mathcal{H}^{(-n)} = H^2_n$ does not contain any non-constant inner function.

When Rudin's book was published in 1980, it was not yet known whether non-constant inner functions existed in the case $n \geq 2$. In fact, Rudin offered the gradient result in his book as evidence against the existence of non-constant inner functions. But in 1982, Aleksandrov and Løw independently succeeded in the construction of non-constant inner functions.

In 2012, Stoll showed that $\mathcal{H}^{(-3/2)}$ contains no non-constant inner functions. (Thanks to Michael Hartz for informing me of this fact.)

We know that the Hardy space $H^2(S) = \mathcal{H}^{(-1)}$ contains all the inner functions, whereas $\mathcal{H}^{(-3/2)}$ contains none, other than the constants.

This comparison raises an interesting question:

For $-3/2 < t < -1$, does $\mathcal{H}^{(t)}$ contain any non-constant inner functions?

This seems to be a very challenging problem.

Corona Theorem.

Costea, Sawyer and Wick (2011): The corona theorem holds for the multiplier algebra $\mathcal{M}$ of the Drury-Arveson space. That is, for $g_1, \ldots, g_k \in \mathcal{M}$, if there is a $c > 0$ such that

$$|g_1(\zeta)| + \cdots + |g_k(\zeta)| \geq c$$

for every $\zeta \in \mathbf{B}$, then there exist $f_1, \ldots, f_k \in \mathcal{M}$ such that

$$f_1 g_1 + \cdots + f_k g_k = 1.$$

This is the first result that truly deserves the appellation "corona theorem" in the multi-variable setting.

Carleson Measure.

A regular Borel measure $d\mu$ on $\mathbf{B}$ is said to be an Carleson measure for the Drury-Arveson space H_n^2 if there is a constant C such that

$$\int |h(\zeta)|^2 d\mu(\zeta) \leq C\|h\|^2$$

for every $h \in H_n^2$. In 2008, Arcozzi, Rochberg and Sawyer gave a characterization for all the Carleson measures for H_n^2.

But unfortunately, this characterization is quite complicated, which can be interpreted as a reflection of the structure of H_n^2.

Essential Norm.

The essential norm of an operator A on a Hilbert space $\mathcal{H}$ is defined by

$$\|A\|_{\text{ess}} = \inf\{\|A+K\| : K \text{ is compact on } \mathcal{H}\}.$$

Despite the fact that, when $n \geq 2$, $\|p\|_\infty$ does not dominate the multiplier norm $\|p\|_{\mathcal{M}} = \|M_p\|$, Arveson showed in his 1998 paper that the identity

$$\|M_p\|_{\text{ess}} = \|p\|_\infty$$

holds for every $p \in \mathbf{C}[z_1, \ldots, z_n]$. Nevertheless:

Fang and X. (2011): For multipliers $f \in \mathcal{M}$ in general, $\|f\|_\infty$ does not dominate the essential norm $\|M_f\|_{\text{ess}}$ on H^2_n if $n \geq 2$. That is, there is NO $0 < C < \infty$ such that

$$\|M_f\|_{\text{ess}} \leq C\|f\|_\infty \quad \text{for every} \quad f \in \mathcal{M}.$$

Essential Commutativity.

Arveson (1998): For all $i, j \in \{1, \dots, n\}$, the commutators $[M_{\zeta_i}, M^*_{\zeta_j}]$ are compact on H^2_n. In fact, these operators belong to the Schatten class $\mathcal{C}_p$ for $p > n$.

The corresponding result for the Hardy space is well known. So perhaps one can interpret the above as saying that H^2_n retains some similarity to the Hardy space. Moreover, we have

Fang and X. (2011): For every $j \in \{1, \dots, n\}$ and every $f \in \mathcal{M}$, the commutator $[M_f, M^*_{\zeta_j}]$ on H^2_n belongs to the Schatten class $\mathcal{C}_p$ for $p > 2n$. Furthermore, for each $p > 2n$, there is a $0 < C(p) < \infty$ such that

$$\|[M_f, M^*_{\zeta_j}]\|_p \leq C(p)\|f\|_{\mathcal{M}}$$

for every multiplier $f \in \mathcal{M}$ and every $j \in \{1, \dots, n\}$, where $\| \cdot \|_p$ is the Schatten p-norm.

Recall that for each $1 \leq p < \infty$, the Schatten class $\mathcal{C}_p$ consists of operators A satisfying the condition

$$\|A\|_p = \left\{ \text{tr}((A^*A)^{p/2}) \right\}^{1/p} < \infty.$$

If $A \in \mathcal{C}_p$, then A is compact.

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To obtain our Schatten-class result mentioned above, we had to use a number of new techniques to overcome the difficulty that there is no measure or L^2 associated with the Drury-Arveson space. In particular, we had to use the fact that even though the tuple $(M_{\zeta_1}, \cdots, M_{\zeta_n})$ is not jointly subnormal, each individual M_{ζ_j} actually is subnormal on H_n^2.

Essential Normality of Submodules.

For each $-n \leq t < \infty$ the Hilbert space $\mathcal{H}^{(t)}$ is also a Hilbert module over the polynomial ring $\mathbf{C}[z_1, \ldots, z_n]$.

In this context, a submodule is a closed linear subspace of $\mathcal{H}^{(t)}$ that is invariant under the multiplication by polynomials.

A submodule Σ of $\mathcal{H}^{(t)}$ is said to be essentially normal if the restricted operators

$$Z_{\Sigma,j} = M_{\zeta_j}|\Sigma, \quad j = 1, \ldots, n,$$

have the property that all commutators $[Z_{\Sigma,i}, Z^*_{\Sigma,j}]$, $1 \leq i, j \leq n$, are compact.

A submodule Σ of $\mathcal{H}^{(t)}$ is said to be p-essentially normal if all commutators $[Z_{\Sigma,i}, Z^*_{\Sigma,j}]$, $1 \leq i, j \leq n$, belong to the Schatten class $\mathcal{C}_p$.

Arveson Conjecture (circa 2000): Every graded submodule of $H_n^2 \otimes \mathbf{C}^m$ is essentially normal.

Graded: the submodule has an orthogonal decomposition in terms of degree. But the problem becomes much more interesting and challenging for submodules that have NO such orthogonal decomposition.

Arveson first verified his conjecture in the case where the submodule is generated by a finite set of monomials (2005).

Later in 2006, Douglas proposed a similar but more refined conjecture for submodules of the Bergman module $\mathcal{H}^{(0)} = L^2(\mathbf{B}, dv)$.

Guo and Wang (2008): In the case $n = 2, 3$, every graded submodule of H_n^2 is p-essentially normal for $p > n$. For arbitrary n, every submodule of H_n^2 generated by a single homogeneous polynomial is p-essentially normal for $p > n$.

Then, came something of a surprise:

Douglas and Wang (2011): For every $q \in \mathbf{C}[z_1, \ldots, z_n]$, the submodule of the Bgerman module $L^2(\mathbf{B}, dv) = \mathcal{H}^{(0)}$ generated by q is p-essentially normal for $p > n$.

The emphasis of the Douglas-Wang result is on the arbitrariness of $q \in \mathbf{C}[z_1, \ldots, z_n]$. This takes us outside the realm of graded submodules and requires analysis that is truly non-trivial. This is an unconditional result in the sense that it makes no assumption about the polynomial q.

Fang and X. (2013): Suppose that $-2 < t < \infty$. Then for every $q \in \mathbf{C}[z_1, \ldots, z_n]$, the submodule of $\mathcal{H}^{(t)}$ generated by q is p-essentially normal for $p > n$.

In particular, if we apply this to the case $t = -1$, we see that for every $q \in \mathbf{C}[z_1, \ldots, z_n]$, the submodule of the Hardy module $\mathcal{H}^{(-1)}$ $= H^2(S)$ generated by q is p-essentially normal for $p > n$.

The Hardy-space problem was specifically mentioned in the Douglas-Wang paper as a challenge, and our 2013 paper was motivated by this. But more interesting is the range of weights $-2 < t < -1$, for which spaces start to mimic the Drury-Arveson space.

Fang and X. (2018): Suppose that $-3 < t \leq -2$. Then for every $q \in \mathbf{C}[z_1, \dots, z_n]$, the submodule of $\mathcal{H}^{(t)}$ generated by q is p-essentially normal for $p > n$.

The consequence of this is that the range $-3 < t \leq -2$ gives us the first unconditional essential normality in the case of Drury-Arveson space. Note that if $n = 2$, then $\mathcal{H}^{(-2)} = H_2^2$, the two-variable Drury-Arveson space. Thus we have

Corollary: For every $q \in \mathbf{C}[z_1, z_2]$, the submodule of H_2^2 generated by q is p-essentially normal for $p > 2$.

But the case $t = -3$, which would give us the three-variable Drury-Arveson space H_3^2, is a major hurdle that is yet to be overcome ...

More generally, we would like to extend our results to the full weight range $t \geq -n$ for arbitrary n. But this seems beyond reach at the moment.

Characterization of Multipliers.

Suppose that $n \geq 2$.

A big mystery in the theory of Drury-Arveson space is the membership in $\mathcal{M}$, the algebra of multipliers. So far, there is no good characterization of the membership $f \in \mathcal{M}$.

In his 1998 paper, Arveson showed that for an analytic function h on $\mathbf{B}$, the finiteness of $\|h\|_\infty$ does not guarantee $h \in H_n^2$. More generally, for $h \in H_n^2 \cap H^\infty$, the norm $\|h\|_\infty$ does not dominate the H_n^2-norm of h.

But, does the membership $f \in H_n^2 \cap H^\infty$ imply $f \in \mathcal{M}$?

Let $k \in \mathbf{N}$ be such that $2k \geq n$. Then given any $f \in H_n^2$, one can define the measure $d\mu_f$ on $\mathbf{B}$ by the formula

$$(***) \qquad d\mu_f(z) = |(R^k f)(z)|^2 (1 - |z|^2)^{2k-n} dv(z),$$

where $R = z_1\partial_1 + \cdots + z_n\partial_n$, the radial derivative, and dv is the normalized volume measure on $\mathbf{B}$.

In 2000, Ortega and Fàbrega showed that $f \in \mathcal{M}$ if and only if $d\mu_f$ is a Carleson measure for H_n^2.

Recall that in 2008, Arcozzi, Rochberg and Sawyer characterized all Carleson measures for H_n^2.

So the combination of the result of Arcozzi, Rochberg and Sawyer and the result of Ortega and Fàbrega is *a* characterization of the membership $f \in \mathcal{M}$. But this characterization is quite complicated, because the condition for a general $d\mu$ to be a Carleson measure for H_n^2 is quite complicated.

But if we are only interested in multipliers, then we are only interested in the $d\mu_f$ given by $(***)$, not the general $d\mu$ on $\mathbf{B}$.

So the question is, for the subclass of measures $d\mu_f$ given by $(***)$, is there a simpler, or more direct, condition that determines when it is a Carleson measure for H_n^2?

Equivalently, is there a simpler, or more direct, characterization of the membership $f \in \mathcal{M}$?

Since the Drury-Arveson space is a reproducing-kernel Hilbert space, it is natural to turn to the reproducing kernel for possible answers. The normalized reproducing kernel for H^2_n is given by

$$k_z(\zeta) = \frac{(1 - |z|^2)^{1/2}}{1 - \langle \zeta, z \rangle},$$

$z, \zeta \in \mathbf{B}$. One of the frequent tools in the study of reproducing-kernel Hilbert spaces is the Berezin transform. But for any $f \in H^2_n$, the Berezin transform

$$\langle f k_z, k_z \rangle$$

is none other than $f(z)$ itself, whose boundedness on $\mathbf{B}$ is not sufficient for the membership $f \in \mathcal{M}$.

But what about something stronger than the Berezin transform? For example, anyone who gives any thought about the multipliers is likely to come up with the following:

Question 1.1. For $f \in H^2_n$, does the condition

$$\sup_{|z|<1} \|fk_z\| < \infty$$

imply the membership $f \in \mathcal{M}$?

Prima facie, one would think that there is at least a fair chance that the answer to Question 1.1 might be affirmative. And that was what we thought for quite a while. What makes this question particularly tempting is that an affirmative answer would give a very simple characterization of the membership $f \in \mathcal{M}$. But that would be too simple a characterization, as it turns out. After a long struggle, we have finally arrived at the conclusion that, tempting though the question may be, its answer is actually negative.

Fang and X. (2015): There exists an $f \in H^2_n$ satisfying the conditions $f \notin \mathcal{M}$ and

$$\sup_{|z|<1} \|fk_z\| < \infty.$$

A consequence of this is that in general, the membership $f \in H^2_n \cap H^\infty$ does not imply $f \in \mathcal{M}$.

Later, a different condition kind of condition for determining the membership $f \in \mathcal{M}$ was examined, one that is in terms of the unnormalized reproducing kernel K_z:

Aleman, Hartz, McCarthy and Richter (2018): For $f \in H_n^2$, the condition

$$\sup_{|z|<1} \mathrm{Re}\langle f, K_z f\rangle < \infty,$$

implies $f \in \mathcal{M}$.

Thus a natural question arises: is the above condition a necessary condition for the membership $f \in \mathcal{M}$. In the same 2018 paper, Aleman et al showed that on the Dirichlet space D_α, $0 < \alpha < 1$, on the unit disc in $\mathbf{C}$, the analogue of the above condition is not necessary for the multipliers of D_α. But it would be nice to know the answer for the Drury-Arveson space H_n^2.

Fang and X. (2020): The function

$$\varphi(\zeta) = \frac{\zeta_2}{\sqrt{1-\zeta_1}},$$

$\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{B}$, is a multiplier of the Drury-Arveson space H^2_n. Moreover, there is a constant $c > 0$ such that

$$\sup_{|z|=r} \mathrm{Re}\langle \varphi, K_z \varphi \rangle \geq c\left(1 + \log \frac{1}{1-r}\right)$$

for every $0 \leq r < 1$. In particular,

$$\sup_{|z|<1} \mathrm{Re}\langle \varphi, K_z \varphi \rangle = \infty.$$

After the publication of our 2020 paper, Aleman, Hartz, McCarthy and Richter also produced $f \in \mathcal{M}$ with

$$\sup_{|z|<1} \operatorname{Re}\langle f, K_z f\rangle = \infty.$$

Their main idea is to embed the multiplier algebra of $D_{1/2}$ into $\mathcal{M}$.

In essence, they showed that the negative answer for the multipliers of D_α in their 2018 paper can be transformed into a negative answer for $\mathcal{M}$.

But the explicit growth rate

$$\sup_{|z|=r} \mathrm{Re}\langle \varphi, K_z\varphi\rangle \geq c\left(1+\log\frac{1}{1-r}\right), \quad 0<r<1,$$

for

$$\varphi(\zeta) = \frac{\zeta_2}{\sqrt{1-\zeta_1}}$$

leads to more:

Fang and X. (2020): Let $\mathcal{F}$ denote the collection of $f \in \mathcal{M}$ satisfying the condition

$$\sup_{|z|<1} \mathrm{Re}\langle f, K_z f\rangle < \infty.$$

With respect to the multiplier norm $\|\cdot\|_{\mathcal{M}}$, $\mathcal{F}$ is nowhere dense in $\mathcal{M}$.

The Reciprocal Problem.

This is a really elementary problem, but one to which we do not have a general answer. This illustrates how little we know about the Drury-Arveson space.

Problem. Let $f \in H^2_n$. Suppose that there is a $c > 0$ such that $|f(\zeta)| \geq c$ for every $\zeta \in \mathbf{B}$. Does it follow that $1/f \in H^2_n$?

The answer is "yes" for $n = 2, 3$ by an argument due to Ricther and Sundberg.

But the problem is completely open for $n \geq 4$.

Submodules

Arveson (2000): Let N be a closed linear subspace of H_n^2 that is invariant under the multiplication by the polynomials. If $N \neq \{0\}$, then N contains a nonzero multiplier of H_n^2.

An immediate consequence of this is that for invariant subspaces N_1, N_2 of H_n^2, if $N_1 \neq \{0\}$ and $N_2 \neq \{0\}$, then $N_1 \cap N_2 \neq \{0\}$.

In the jargon of invariant-subspace theory, this fact says that there are no non-trivial invariant subspaces of H_n^2 that are disjoint.

This is definitely not the case for some of the other reproducing-kernel Hilbert spaces. For example, one can easily construct invariant subspaces N_1, N_2 of the Bergman space $L_a^2(\mathbf{B}, dv)$ such that $N_1 \cap N_2 = \{0\}$ while $N_1 \neq \{0\}$ and $N_2 \neq \{0\}$. In particular, this implies that $N_1 \cap H^\infty(\mathbf{B}) = \{0\}$ and $N_2 \cap H^\infty(\mathbf{B}) = \{0\}$.

What about the Hardy space?

This result of Arveson can also be proved using Drury's ideas.

Recall that Drury's proof of the von Neumann inequality for row contraction $(A_1, \ldots, A_n)$ is the identity

$$(\#) \qquad Zp(A_1^*, \ldots, A_n^*) = (p(M_{\zeta_1}^*, \ldots, M_{\zeta_n}^*) \otimes 1)Z$$

for $p \in \mathbf{C}[z_1, \ldots, z_n]$, where

$$(Zx)(\zeta) = \sum_{\alpha \in \mathbf{Z}_+^n} \frac{|\alpha|!}{\alpha!}(1 - A_1A_1^* - \cdots - A_nA_n^*)^{1/2} A^{*\alpha} x \zeta^\alpha,$$

$x \in \mathcal{H}$, is an isometry from $\mathcal{H}$ to $H_n^2 \otimes \mathcal{H}$.

Now if we take adjoints on both sides of (#), we get

$$(\#\#) \qquad p(A_1, \ldots, A_n)Z^* = Z^*(p(M_{\zeta_1}, \ldots, M_{\zeta_n}) \otimes 1)$$

for every $p \in \mathbf{C}[z_1, \ldots, z_n]$.

Let N be a submodule (i.e., an invariant subspace) of H_n^2. Consider the Z for the case where $\mathcal{H} = N$ and

$$(A_1, \ldots, A_n) = (M_{\zeta_1}|N, \ldots, M_{\zeta_n}|N).$$

For each $y \in N$, set $\varphi_y = Z^*(1 \otimes y) \in N$. Then (##) gives us

$$\|p\varphi_y\|_{H_n^2} = \|p\varphi_y\|_N \le \|p \otimes y\|_{H_n^2 \otimes N} = \|p\|_{H_n^2}\|y\|_N$$

for every $p \in \mathbf{C}[z_1, \ldots, z_n]$. This means that $\varphi_y \in \mathcal{M}$. Finally, since the linear span of all $p\varphi_y = Z^*(p \otimes y)$ is dense in N, the condition $N \neq \{0\}$ implies that $\varphi_y \neq 0$ for some $y \in N$.

Based on this proof, we can say that Arveson's result that $N \neq \{0\}$ implies $N \cap \mathcal{M} \neq \{0\}$ for any submodule N of H^2_n is actually a "close cousin" of the von Neumann inequality for row contractions.

Note that when $n = 1$, this result is a consequence of Beurling's theorem. So one say that this one particular aspect of Beurling's theorem is retained by H^2_n for all $n \geq 2$.

Thank you!