Commutant lifting and Nevanlinna-Pick interpolation in several variables

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Notation

1. $\mathscr{H}$, $\mathscr{E}$, $\mathscr{E}_*$, etc. : Hilbert spaces (separable and over $\mathbb{C}$).
2. Linear operators are bounded: $T \in \mathscr{B}(\mathscr{H})$.
3. $\mathbb{D}^n = \{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \ldots, n\}$: open polydisc.
4. $\mathbb{B}^n = \{\boldsymbol{z} \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^2 < 1\}$: open ball.
5. $H^\infty(\Omega) = \{\varphi \in \mathscr{O}(\Omega) : \|\varphi\| := \sup_{z\in\Omega} |\varphi(z)| < \infty\}$. Here $\Omega = \mathbb{B}^n$ or $\mathbb{D}^n$.
6. $\mathscr{S}(\Omega) = \{\varphi \in H^\infty(\Omega) : \|\varphi\| \leq 1\}$.
7. $P_{\mathscr{S}}$ = orthogonal projection onto $\mathscr{S}$, where $\mathscr{S} \subseteq \mathscr{H}$ is a closed subspace.
8. $i_{\mathscr{S}} : \mathscr{S} \hookrightarrow \mathscr{H}$ is the inclusion map.
9. RKHS = Reproducing kernel Hilbert space.

Road map

1. The classical Nevanlinna–Pick interpolation theorem
2. Sarason's commutant lifting theorem
3. Dilations
4. A halt at the bidisc
5. Commutant lifting theorem
6. Factorizations, and
7. Nevanlinna-Pick interpolation on $\mathbb{B}^n$.

Nevanlinna–Pick interpolation

Theorem 1 (Nevanlinna (1919) and Pick (1916))

Given distinct p points $\{z_i\}_{i=1}^{p} \subseteq \mathbb{D}$ and p points $\{w_i\}_{i=1}^{p} \subseteq \mathbb{D}$, there exists a $\varphi \in \mathscr{S}(\mathbb{D})$ such that

$$\varphi(z_i) = w_i \qquad (i = 1, \dots, p),$$

if and only if the Pick matrix

$$\left(\frac{1-w_i\bar{w}_j}{1-z_i\bar{z}_j}\right)_{p\times p} \geq 0.$$

$$\boxed{\exists \varphi \in \mathscr{S}(\mathbb{D}) \text{ s.t } \varphi(z_i) = w_i} \rightleftarrows \boxed{\left(\frac{1-w_i\bar{w}_j}{1-z_i\bar{z}_j}\right)_{p\times p} \geq 0}$$

NP interpolation and Sarason

$$\boxed{\exists \varphi \in \mathscr{S}(\mathbb{D}) \text{ s.t } \varphi(z_i) = w_i} \rightleftarrows \boxed{\left(\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j}\right)_{p \times p} \geq 0}$$

Complex Analysis

Linear Algebra.

↑

$$\boxed{\text{Commutant lifting theorem}}$$

Operator Theory.

Note: See the article by J. McCarthy. Pick's theorem—what's the big deal? 2003.

CLT: Hardy space

The Hardy space over $\mathbb{D}$:

$$H^2(\mathbb{D}) := \{f = \sum_{m=0}^{\infty} a_m z^m \in \mathscr{O}(\mathbb{D}) : \|f\| = (\sum_{m=0}^{\infty} |a_m|^2)^{\frac{1}{2}} < \infty\}.$$

And, the shift M_z on $H^2(\mathbb{D})$ is the multiplication operator

$$M_z f = zf \qquad (f \in H^2(\mathbb{D})).$$

Note:

1. S on $l^2(\mathbb{Z}_+) \cong M_z$ on $H^2(\mathbb{D})$.
2. $\{M_z\}' = \{M_\varphi : \varphi \in H^\infty(\mathbb{D})\}$.
3. $H^2(\mathbb{D})$ is a RKHS corresponding to the Szegö kernel

$$s(z,w) = (1 - z\bar{w})^{-1} \qquad (z, w \in \mathbb{D}).$$

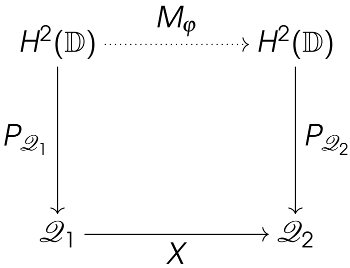
4. $M_\varphi^* s(\cdot, w) = \overline{\varphi(w)} s(\cdot, w)$ for all $\varphi \in H^\infty(\mathbb{D})$ and $w \in \mathbb{D}$.

Theorem 2 (Sarason, 1967)

Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are M_z^*-invariant closed subspaces of $H^2(\mathbb{D})$. Let $X \in \mathscr{B}(\mathcal{Q}_1, \mathcal{Q}_2)$ and $\|X\| \leq 1$. If $X(P_{\mathcal{Q}_1} M_z|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2} M_z|_{\mathcal{Q}_2})X$, then there exists a $\varphi \in \mathscr{S}(\mathbb{D})$ such that

$$X = P_{\mathcal{Q}_2} M_\varphi|_{\mathcal{Q}_1}.$$

That is, we have the following commutative diagram:



Sarason's CLT yields a simple and elegant proof of the interpolation theorem.

The easy direction: If $\varphi(z_i) = w_i$ for $\varphi \in \mathscr{S}(\mathbb{D})$, then $\|M_\varphi\| \leq 1$ and $M_\varphi^* s(\cdot, w) = \overline{\varphi(w)} s(\cdot, w)$, $w \in \mathbb{D} \Rightarrow$ the Pick matrix ≥ 0.

The not so easy direction: Suppose the Pick matrix ≥ 0. Then one proceeds as follows:

1. Consider the M_z^*-invariant finite dimensional subspace
$$\mathscr{Q} = \text{span}\,\{s(\cdot, z_i) : i = 1, \ldots, p\}.$$
2. Define $X : \mathscr{Q} \to \mathscr{Q}$ by $Xs(\cdot, z_i) = \bar{w}_i s(\cdot, z_i)$, $i = 1, \ldots, p$.
3. Positivity of the Pick matrix $\Rightarrow$ X is a contraction.
4. On the other hand, $X^*(P_{\mathscr{Q}} M_z|_{\mathscr{Q}}) = (P_{\mathscr{Q}} M_z|_{\mathscr{Q}}) X^*$.
5. Apply the CLT to get a solution $\varphi \in \mathscr{S}(\mathbb{D})$.

Interconnections

1. Commutant lifting theorem
2. Nevanlinna–Pick interpolation
3. Dilations
4. von Neumann inequality
5. Invariant subspaces
6. Toeplitz corona (if not corona) theorem, etc.

Groundwork

Drury-Arveson space

Definition 3

The Drury-Arveson space H_n^2 is a RKHS corresponding to the kernel

$$k_1(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-1} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Remark 4

$(M_{z_1}, \ldots, M_{z_n})$ *defines a row contraction on* H_n^2*, that is*

$$\sum_{i=1}^{n} M_{z_i} M_{z_i}^* \leq I_{H_n^2}.$$

Definition 5

Let $k : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be a kernel such that k is analytic in the first variables $\{z_1, \ldots, z_n\}$. We say that k is regular if there exists a kernel $\tilde{k}$ on $\mathbb{B}^n$ such that

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

The RKHS $\mathscr{H}_k \subseteq \mathscr{O}(\mathbb{B}^n)$ corresponding to the kernel k will be referred as a regular RKHS.

The above factorization essentially implies that $(M_{z_1}, \ldots, M_{z_n})$ defines a commuting row contraction on $\mathscr{H}_k$ (cf. Douglas - Misra - S. 2012):

$$\sum_{i=1}^{n} M_{z_i} M_{z_i}^* \leq I_{\mathscr{H}_k}.$$

Example 6

For each $\lambda > 1$, $\mathscr{H}_{k_\lambda}$ is a regular RKHS, where

$$k_\lambda(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-\lambda} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Note that $\mathscr{H}_{k_\lambda}$ is the Hardy/Bergman/weighted Bergman space for $\lambda = n$, $n+1$, and $n+1+\alpha$ for any $\alpha > 0$, respectively.

Vector-valued spaces: If $\mathscr{E}$ is a Hilbert space and k is a regular kernel, then we simply say that $\mathscr{H}_k \otimes \mathscr{E}$ is a (vector-valued) regular RKHS.

Dilations

Definition 7

Suppose $T = (T_1, \ldots, T_n)$ is a commuting tuple on $\mathscr{H}$. We say that T dilates to $(M_{z_1} \otimes I_{\mathscr{E}_*}, \ldots, M_{z_n} \otimes I_{\mathscr{E}_*})$ on $H^2_n \otimes \mathscr{E}_*$ if there exists an isometry $\Pi : \mathscr{H} \to H^2_n \otimes \mathscr{E}_*$ such that

$$\Pi T_i^* = (M_{z_i} \otimes I_{\mathscr{E}_*})^* \Pi \qquad (i = 1, \ldots, n).$$

We often say that $\Pi : \mathscr{H} \to H^2_n \otimes \mathscr{E}_*$ is a *dilation* of T.

From now on: We assume that $\mathscr{H}_k$ is a regular RKHS.

Dilations

Theorem 8 (Muller-Vasilescu, Arveson, Athavale, Douglas-Misra-S, Kumari-Sarkar-S-Timotin)

$(M_{z_1} \otimes I_{\mathscr{E}}, \ldots, M_{z_n} \otimes I_{\mathscr{E}})$ on $\mathscr{H}_k \otimes \mathscr{E}$ dilates to $(M_{z_1} \otimes I_{\mathscr{E}_*}, \ldots, M_{z_n} \otimes I_{\mathscr{E}_*})$ on $H_n^2 \otimes \mathscr{E}_*$ for some Hilbert space $\mathscr{E}_*$.

More specifically: There exist a Hilbert space $\mathscr{E}_*$ and an isometry

$$\Pi_k : \mathscr{H}_k \otimes \mathscr{E} \to H_n^2 \otimes \mathscr{E}_*,$$

such that

$$\Pi_k (M_{z_i} \otimes I_{\mathscr{E}})^* = (M_{z_i} \otimes I_{\mathscr{E}_*})^* \Pi_k \qquad (i = 1, \ldots, n).$$

Multipliers

Definition 9

Given regular RKHSs $\mathscr{H}_{k_1} \otimes \mathscr{E}_1$ and $\mathscr{H}_{k_2} \otimes \mathscr{E}_2$, a function $\Phi : \mathbb{B}^n \to \mathscr{B}(\mathscr{E}_1, \mathscr{E}_2)$ is called a *multiplier* from $\mathscr{H}_{k_1} \otimes \mathscr{E}_1$ to $\mathscr{H}_{k_2} \otimes \mathscr{E}_2$ if

$$\Phi(\mathscr{H}_{k_1} \otimes \mathscr{E}_1) \subseteq \mathscr{H}_{k_2} \otimes \mathscr{E}_2.$$

In this case, $M_\Phi f = \Phi f$ defines a bounded linear operator from $\mathscr{H}_{k_1} \otimes \mathscr{E}_1$ to $\mathscr{H}_{k_2} \otimes \mathscr{E}_2$.

The multiplier space $\mathscr{M}(\mathscr{H}_{k_1} \otimes \mathscr{E}_1, \mathscr{H}_{k_2} \otimes \mathscr{E}_2)$ is the set of all multipliers from $\mathscr{H}_{k_1} \otimes \mathscr{E}_1$ to $\mathscr{H}_{k_2} \otimes \mathscr{E}_2$.

Theorem 10 (cf. S: 2016)

Suppose $X \in \mathscr{B}(H_n^2 \otimes \mathcal{E}_1, \mathscr{H}_k \otimes \mathcal{E}_2)$. *Then*

$$X(M_{z_i} \otimes I_{\mathcal{E}_1}) = (M_{z_i} \otimes I_{\mathcal{E}_2})X \qquad (i = 1, \ldots, n).$$

if and only if $X = M_\Phi$ *for some* $\Phi \in \mathscr{M}(H_n^2 \otimes \mathcal{E}_1, \mathscr{H}_k \otimes \mathcal{E}_2)$.

Proof.

Consider $(M_{z_i}^* \otimes I_{\mathcal{E}_1})X^* = X^*(M_{z_i} \otimes I_{\mathcal{E}_2})^*$, and use the fact that

$$\cap_{i=1}^n \ker(M_{z_i} - w_i I_{H_n^2})^* = \mathbb{C}k_1(\cdot, w) \qquad (\boldsymbol{w} \in \mathbb{B}^n).$$

□

Denote by

$$\mathscr{M}_1(H_n^2 \otimes \mathcal{E}_1, \mathscr{H}_k \otimes \mathcal{E}_2) = \{\Phi \in \mathscr{M}(H_n^2 \otimes \mathcal{E}_1, \mathscr{H}_k \otimes \mathcal{E}_2) : \|M_\Phi\| \leq 1\}.$$

CLT on H^2_n

Definition 11

A closed subspace $\mathcal{Q} \subseteq \mathcal{H}_k \otimes \mathcal{E}$ is said to be quotient module if $(M_{z_i} \otimes I_{\mathcal{E}})^* \mathcal{Q} \subseteq \mathcal{Q}$, $i = 1, \ldots, n$.

Theorem 12 (Ambrozie-Timotin, Ball-Trent-Vinnikov)

Suppose $\mathcal{Q}_1 \subseteq H^2_n \otimes \mathcal{E}_1$ *and* $\mathcal{Q}_2 \subseteq H^2_n \otimes \mathcal{E}_2$ *be quotient modules, and* $X \in \mathcal{B}(\mathcal{Q}_1, \mathcal{Q}_2)$ *be a contraction. If*

$$X(P_{\mathcal{Q}_1}(M_{z_i} \otimes I_{\mathcal{E}_1})|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2}(M_{z_i} \otimes I_{\mathcal{E}_2})|_{\mathcal{Q}_2})X \qquad (i = 1, \ldots, n),$$

then there exists $\Phi \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, H^2_n \otimes \mathcal{E}_2)$ *such that* $P_{\mathcal{Q}_2} M_\Phi|_{\mathcal{Q}_1} = X$.

Also see: Davidson and Le, Commutant lifting for commuting row contractions. 2010.

The bidisc

Theorem 13 (Agler)

Given p points $\{(\alpha_i, \beta_i)\}_{i=1}^{p} \subseteq \mathbb{D}^2$ *and p points* $\{w_i\}_{i=1}^{p} \subseteq \mathbb{D}$*, there exists a* $\varphi \in \mathscr{S}(\mathbb{D}^2)$ *such that*

$$\varphi((\alpha_i, \beta_i)) = w_i \qquad (i = 1, \ldots, p),$$

if and only if there exist positive $p \times p$ *matrices* Γ *and* Δ *such that*

$$(1 - \bar{w}_i w_j) = (1 - \bar{\alpha}_i \alpha_j)\Gamma_{ij} + (1 - \bar{\beta}_i \beta_j)\Delta_{ij}.$$

Note 1: $\mathbb{D}^n$, $n \geq 2$, is another story and with different kinds of challenges (and surprises)! (However, see: Kosiński and Zwonek, Nevanlinna-Pick interpolation problem in the ball. 2018.)

Note 2: See the paper: J. McCarthy, Shining a Hilbertian lamp on the bidisk. 2012.

On today's agenda

Dilations and a variation of the H^2_n-CLT

The following is a variation, in terms of dilations, of the H^2_n-CLT (Theorem 12).

Theorem 14

Suppose $T = (T_1, \ldots, T_n)$ *and* $\hat{T} = (\hat{T}_1, \ldots, \hat{T}_n)$ *are commuting tuples on* $\mathscr{H}$ *and* $\hat{\mathscr{H}}$*, respectively,* $X \in \mathscr{B}(\mathscr{H}, \hat{\mathscr{H}})$*,* $\|X\| \leq 1$*, and*

$$XT_i = \hat{T}_i X \qquad (i = 1, \ldots, n).$$

If $\Pi : \mathscr{H} \to H^2_n \otimes \mathscr{E}$ *and* $\hat{\Pi} : \hat{\mathscr{H}} \to H^2_n \otimes \hat{\mathscr{E}}$ *are dilations of* T *and* $\hat{T}$*, respectively, then there exists* $\Phi \in \mathscr{M}_1(H^2_n \otimes \mathscr{E}, H^2_n \otimes \hat{\mathscr{E}})$ *such that*

$$X = \hat{\Pi}^* M_\Phi \Pi.$$

Proof: We need to chase the following commutative diagram:

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{\quad \Pi \quad} & H_n^2 \otimes \mathscr{E} \\ {\scriptstyle X}\downarrow & & \vdots\, {\scriptstyle M_\Phi} \\ \hat{\mathscr{H}} & \xrightarrow[\quad \hat{\Pi} \quad]{} & H_n^2 \otimes \hat{\mathscr{E}} \end{array}$$

Consider the quotient modules $\mathscr{Q} := \Pi \mathscr{H} \subseteq H_n^2 \otimes \mathscr{E}$ and $\hat{\mathscr{Q}} := \hat{\Pi} \hat{\mathscr{H}} \subseteq H_n^2 \otimes \mathscr{E}_*$. If

$$\tilde{X} = \hat{\Pi} X \Pi^*|_{\mathscr{Q}},$$

then a little computation shows that $\tilde{X} \in \mathscr{B}(\mathscr{Q}, \hat{\mathscr{Q}})$ and

$$\tilde{X}(P_{\mathscr{Q}}(M_{z_i} \otimes I_{\mathscr{E}})|_{\mathscr{Q}}) = (P_{\hat{\mathscr{Q}}}(M_{z_i} \otimes I_{\hat{\mathscr{E}}})|_{\hat{\mathscr{Q}}})\tilde{X} \qquad (i = 1, \ldots, n).$$

It then follows from the H^2_n-CLT that

$$\tilde{X} = P_{\hat{\mathscr{Q}}} M_\Phi|_{\mathscr{Q}},$$

for some $\Phi \in \mathscr{M}_1(H^2_n \otimes \mathscr{E}, H^2_n \otimes \hat{\mathscr{E}})$. By a repeated application of $P_{\mathscr{Q}} = \Pi\Pi^*$ and $\mathscr{Q} = \text{ran}\, \Pi = \text{ran}\, \Pi\Pi^*$, it follows that $X = \hat{\Pi}^* M_\Phi \Pi$. $\square$

Prologue (to the CLT)

Definition 15 (Dilations of $\mathscr{Q}$)

Let $\mathscr{Q} \subseteq \mathscr{H}_k \otimes \mathscr{E}$ be a quotient module. An isometry $\Pi : \mathscr{Q} \to H^2_n \otimes \mathscr{E}_*$ is said to be a dilation of $\mathscr{Q}$ if Π is a dilation of $(P_{\mathscr{Q}} M_{z_1}|_{\mathscr{Q}}, \ldots, P_{\mathscr{Q}} M_{z_n}|_{\mathscr{Q}})$.

Lemma 16

Let $\mathscr{Q} \subseteq \mathscr{H}_k \otimes \mathscr{E}$ be a quotient module. If $\Pi : \mathscr{H}_k \otimes \mathscr{E} \to H^2_n \otimes \mathscr{E}_*$ is a dilation of $\mathscr{H}_k \otimes \mathscr{E}$, then

$$\Pi_{\mathscr{Q}} := \Pi \circ i_{\mathscr{Q}} : \mathscr{Q} \to H^2_n \otimes \mathscr{E}_*,$$

is a dilation $\mathscr{Q}$.

Proof.

Some elementary computations with projections and inclusions. Essentially, we have the following:

$$\mathcal{Q} \hookrightarrow \mathscr{H}_k \otimes \mathcal{E} \xrightarrow{\Pi} H^2_n \otimes \mathcal{E}_*.$$

□

The CLT

Theorem 17

Let $\mathscr{Q}_1 \subseteq H^2_n \otimes \mathscr{E}_1$ and $\mathscr{Q}_2 \subseteq \mathscr{H}_k \otimes \mathscr{E}_2$ be quotient moudles. Let $X \in \mathscr{B}(\mathscr{Q}_1, \mathscr{Q}_2)$, $\|X\| \leq 1$, and let

$$X(P_{\mathscr{Q}_1}(M_{z_i} \otimes I_{\mathscr{E}_1})|_{\mathscr{Q}_1}) = (P_{\mathscr{Q}_2}(M_{z_i} \otimes I_{\mathscr{E}_2})|_{\mathscr{Q}_2})X,$$

for all i. Then there exists $\Phi \in \mathscr{M}_1(H^2_n \otimes \mathscr{E}_1, \mathscr{H}_k \otimes \mathscr{E}_2)$ such that

$$X = P_{\mathscr{Q}_2} M_\Phi|_{\mathscr{Q}_1}.$$

Proof: Clearly, $i_{\mathscr{Q}_1} : \mathscr{Q}_1 \hookrightarrow H^2_n \otimes \mathscr{E}_1$ is a dilation of $\mathscr{Q}_1$. Let $\Pi_k : \mathscr{H}_k \otimes \mathscr{E}_2 \to H^2_n \otimes \hat{\mathscr{E}}$ be a dilation of $\mathscr{H}_k \otimes \mathscr{E}_2$ (assured by Theorem 8). By Lemma 16:

$$\Pi_{\mathscr{Q}_2} = \Pi_k i_{\mathscr{Q}_2} : \mathscr{Q}_2 \to H^2_n \otimes \hat{\mathscr{E}},$$

is a dilation of $\mathscr{Q}_2$. That is:

$$
\begin{array}{ccc}
\mathscr{Q}_1 & \xrightarrow{\;\; i_{\mathscr{Q}_1} \;\;} & H^2_n \otimes \mathscr{E}_1 \\
\Big\downarrow X & & \vdots\, M_{\Phi_1} \\
\mathscr{Q}_2 & \xrightarrow[\;\; \Pi_{\mathscr{Q}_2} \;\;]{} & H^2_n \otimes \hat{\mathscr{E}}
\end{array}
$$

Then Theorem 14 yields $X = \Pi^*_{\mathscr{Q}_2} M_{\Phi_1} i_{\mathscr{Q}_1}$ for some $\Phi_1 \in \mathscr{M}(H^2_n \otimes \mathscr{E}_1, H^2_n \otimes \hat{\mathscr{E}})$. Hence

$$X = i^*_{\mathscr{Q}_2} (\Pi^*_k M_{\Phi_1}) i_{\mathscr{Q}_1}.$$

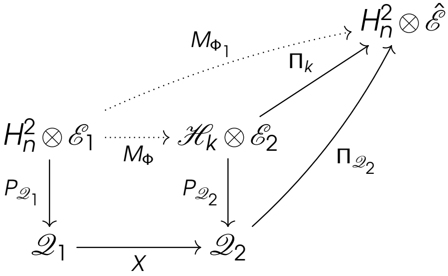
One checks that $\Pi^*_k M_{\Phi_1} : H^2_n \otimes \mathscr{E}_1 \to \mathscr{H}_k \otimes \mathscr{E}_2$ intertwines the shifts, that is

$$\Pi^*_k M_{\Phi_1} = M_\Phi,$$

for some $\Phi \in \mathscr{M}(H^2_n \otimes \mathscr{E}_1, \mathscr{H}_k \otimes \mathscr{E}_2)$. Then $X = i^*_{\mathscr{Q}_2} M_\Phi i_{\mathscr{Q}_1}$, and thus .... $X = P_{\mathscr{Q}_2} M_\Phi |_{\mathscr{Q}_1}$. □

The diagram

That is, the following diagram commutes:



Factorizations - I

Theorem 18

Suppose $\mathscr{H}_{\tilde{k}}$ is the RKHS corresponding to $\tilde{k}$, where $k = k_1\tilde{k}$. If $\mathscr{E}$ is a Hilbert space, then there exists a co-isometric multiplier $\Psi_k \in \mathscr{M}(H^2_n \otimes (\mathscr{H}_{\tilde{k}} \otimes \mathscr{E}), \mathscr{H}_k \otimes \mathscr{E})$ such that

$$k(\boldsymbol{z}, \boldsymbol{w}) I_{\mathscr{E}} = \frac{\Psi_k(\boldsymbol{z})\Psi_k(\boldsymbol{w})^*}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Factorizations - I (Example)

Let $m > 1$ be an integer. Set

$$k_m(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-m} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Then $\tilde{k}_m(\boldsymbol{z}, \boldsymbol{w}) = k_{m-1}(\boldsymbol{z}, \boldsymbol{w})$, and $\Psi_{k_m}(\boldsymbol{w})^* : \mathscr{E} \to \mathscr{H}_{k_{m-1}} \otimes \mathscr{E}$ is given by

$$\Psi_{k_m}(\boldsymbol{w})^* \eta = k_{m-1}(\cdot, \boldsymbol{w}) \otimes \eta,$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$ and $\eta \in \mathscr{E}$.

Factorizations - II

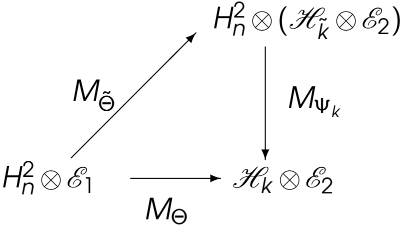
Theorem 19

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, and let $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ be an analytic function. In the setting of Theorem 18, the following conditions are equivalent:
(i) $\Theta \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$,
(ii) there exists $\tilde{\Theta} \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2))$ *such that*

$$M_\Theta = M_{\Psi_k} M_{\tilde{\Theta}}.$$

Also see (the scalar case): Ball - Bolotnikov (2017); and Aleman - Hartz - McCarthy - Richter (2019).

More specifically, the multiplier Ψ_k makes the following diagram commutative:



NP interpolation

Let $\mathscr{B}_1(\mathcal{E}_1, \mathcal{E}_2) = \{A \in \mathscr{B}(\mathcal{E}_1, \mathcal{E}_2) : \|A\| < 1\}$.

Problem: Given $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$, $\{W_i\}_{i=1}^m \subseteq \mathscr{B}_1(\mathcal{E}_1, \mathcal{E}_2)$, find necessary and sufficient conditions for the existence of a multiplier $\Phi \in \mathscr{M}_1(H_n^2 \otimes \mathcal{E}_1, \mathscr{H}_k \otimes \mathcal{E}_2)$ such that

$$\Phi(\boldsymbol{z}_i) = W_i \qquad (i = 1, \ldots, m). \tag{1}$$

Theorem 20

Let $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$ *and* $\{W_i\}_{i=1}^m \subseteq \mathscr{B}_1(\mathscr{E}_1, \mathscr{E}_2)$. *TFAE:*
(i) There exists $\Phi \in \mathscr{M}_1(H_n^2 \otimes \mathscr{E}_1, \mathscr{H}_k \otimes \mathscr{E}_2)$ *such that* $\Phi(\boldsymbol{z}_i) = W_i$, $i = 1, \ldots, m$.
(ii) $\sum_{1 \leq i,j \leq m} \left\langle \left(k(\boldsymbol{z}_i, \boldsymbol{z}_j) I_{\mathscr{E}_2} - \frac{W_i W_j^*}{1 - \langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle} \right) \eta_j, \eta_i \right\rangle$ *for all* $\eta_1, \ldots, \eta_m \in \mathscr{E}_2$.
(iii) There exists $\tilde{\Phi} \in \mathscr{M}_1(H_n^2 \otimes \mathscr{E}_1, H_n^2 \otimes (\mathscr{H}_{\tilde{k}} \otimes \mathscr{E}_2))$ *such that*

$$\Psi_k(\boldsymbol{z}_i)\tilde{\Phi}(\boldsymbol{z}_i) = W_i \qquad (i = 1, \ldots, n).$$

In the case of scalar-valued multipliers:
$(i) \Leftrightarrow (ii)$ is due to Aleman, Hartz, McCarthy and Richter (2019).
In addition, if $n = 1$ and $\tilde{k}(z, w) = (1 - z\bar{w})^{-m}$, $m \in \mathbb{N}$, then $(i) \Leftrightarrow (ii)$ was proved by Ball and Bolotnikov (2017).

NP interpolation - Example

Consider the RKHS $\mathscr{H}_m$, $m > 1$, corresponding to the kernel

$$k_m(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-m} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Then we have the following:
Given distinct m points $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and m points $\{w_i\}_{i=1}^m \subseteq \mathbb{D}$, there exists a $\varphi \in \mathscr{M}_1(H_n^2, \mathscr{H}_m)$ such that $\varphi(\boldsymbol{z}_i) = w_i$, $i = 1, \ldots, m$, if and only if

$$\left(\frac{1}{(1 - \langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle)^m} - \frac{w_i \bar{w}_j}{1 - \langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle} \right)_{m \times m} \geq 0.$$

Thanks!!