Monotonicity, convexity, and realization of noncommutative functions

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Classical motivation

A **Pick function** is an analytic function that maps the complex upper half plane into itself.

Pick functions sit at the heart of a number of interconnected ideas:

- Interpolation
- Analytic continuation
- Integral representation
- Moment problems
- Matrix monotonicity and matrix convexity

Goals of the talk

In this talk, I want to discuss

1. classical results connecting Pick functions with matrix monotonicity and matrix convexity;
2. related generalizations in the setting of noncommutative function theory;
3. thematic ideas from the proofs.

Representation of Pick functions

Let Π denote the complex upper half plane. A **Pick function** is an analytic map $f : \Pi \to \overline{\Pi}$.

Theorem (Nevanlinna's representation)

f is a Pick function if and only if there exist $a \in \mathbb{R}$, $b \geq 0$, and a positive finite Borel measure μ with $\int \frac{1}{t^2+1} d\mu < \infty$ such that for all $z \in \Pi$

$$f(z) = a + bz + \int_{\mathbb{R}} \frac{1}{t-z} - \frac{t}{t^2+1} \, d\mu(t).$$

Moreover, a, b, μ are uniquely determined by f.

It is straightforward to show that the imaginary part of such a representation is positive on the upper halfplane (a fact invoked in continuation arguments).

Connection with positive matrices

The Nevanlinna representation is connected to matrices by way of Pick's well-known observation that for a given Pick function h, for any collection of points $z_1, \ldots, z_n$ in the upper half plane, the so-called Pick matrix is positive semi-definite:

$$\left(\frac{h(z_i) - \overline{h(z_j)}}{z_i - \overline{z}_j} \right)_{i,j} \geq 0$$

(This leads, e.g., to the Nevanlinna-Pick interpolation theorem.)

Matrix inequalities

Let A and B be self-adjoint matrices:

- We say $A \leq B$ if $B - A$ is positive semi-definite.
- We say $A < B$ if $B - A$ is positive definite.

The functional calculus

Let $f : (a, b) \to \mathbb{R}$.
Given a self-adjoint matrix A with spectrum in (a, b) diagonalized by a unitary matrix U, that is,

$$A = U^* \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} U$$

we define the expression $f(A)$ via the following formula.

$$f(A) = U^* \begin{pmatrix} f(\lambda_1) & 0 & \dots \\ 0 & f(\lambda_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} U.$$

Note that under this definition, $f(A \oplus B) = f(A) \oplus f(B)$ and $f(U^*AU) = U^*f(A)U)$ when U is unitary.

Special matrix functions

Let $f : (a, b) \to \mathbb{R}$.

- f is **matrix monotone** if, for any natural number $n \in \mathbb{N}$, and any pair of n by n self-adjoint matrices A and B with spectrum in (a, b),
$$A \leq B \Rightarrow f(A) \leq f(B).$$
- f is **matrix convex** whenever f evaluated on n by n matrices via the matrix functional calculus is a matrix-valued convex function. That is,
$$f\left(\frac{A+B}{2}\right) \leq \frac{f(A)+f(B)}{2}$$
for all A, B with spectrum in (a, b).

Monotone $\neq$ matrix monotone

Matrix monotone and matrix convex functions are strict subsets of the monotone and convex functions respectively.

For example,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \leq \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The function $f(x) = x^3$ is everywhere monotone increasing. However,

$$f\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \text{ and } f\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix},$$

and

$$\begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix} - \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ 4 & 1 \end{pmatrix}$$

which is not positive semi-definite, as the determinant is negative.

Classical representations: Löwner-Nevanlinna

Theorem (Löwner's theorem)

A function $f : (a, b) \to \mathbb{R}$ is matrix monotone if and only if f analytically continues to a Pick function $f : \Pi \cup (a, b) \to \overline{\Pi}$.

One very straightforward proof of Löwner's theorem runs through Nevanlinna's solution of the Hamburger moment problem, which essentially equates the existence of a positive measure with given moments exists with the positivity of a Hankel matrix of moments.

Integral representations of Nevanlinna type play a key role in the proof. (See e.g. Donoghue, Matrix Monotone Functions, Ch 9)

Classical representations: Kraus

Theorem (Kraus 1937)

Let $f : (-1,1) \to \mathbb{R}$*.* f *is matrix convex if and only if*

$$f(x) = a + bx + \int_{[-1,1]} \frac{x^2}{1+tx}\, d\mu(t)$$

where $a, b \in \mathbb{R}$ *and* μ *is a finite measure supported on* $[-1,1]$*. Note that all such functions analytically continue to the upper half plane.*

Kraus showed that matrix convexity implied that f was C^2 and that $f(x)/x$ is matrix monotone on $(-1,1)$. Proof of the theorem above follows from the machinery of the proof of Löwner's theorem.

Thematic ideas in classical proofs

There are many proofs of Löwner's theorem. What makes such an argument amenable to several complex variables?

- Monotonicity/convexity $\Rightarrow$ some family of matrices is positive
- Positive matrices give rise to a measure and representation that is real analytic
- Real analytic structure continues to the upper half plane

Such theorems are essentially automatic analyticity results.

Examples of representations

Question

Does this heuristic generalize?

Lots of realization/representation analogues to classical theorems. Here's a subset.

- Nevanlinna representations (Agler-T.D.-Young '16, Pascoe-T.D. '17, Palfia '20)
- Kraus-type representations (Helton-McCullough-Vinnikov '06, Pascoe-T.D. '20, Palfia-Gaal '20, Palfia '21)
- Partially convex functions (Jury-Klep-Mancuso-McCullough-Pascoe)
- Plurisubharmonic functions (Dym-Helton-Kelp-McCullough-Volcic)
- More!

Evidence that an automatic analyticity approach might be achievable

The matrix universe

Let V be a real vector space. We define the **matrix universe over** V to be:

$$\mathcal{M}(V) = \bigcup M_n(\mathbb{C}) \otimes V.$$

We define the **real matrix universe over** V to be:

$$\mathcal{S}(V) = \bigcup S_n(\mathbb{C}) \otimes V.$$

Given A, B of the same size, we say $A \leq B$ if $B - A$ is positive semidefinite as an element of $S_n(\mathbb{C}) \otimes V$.

The matrix universe (more usual)

We define the **matrix universe** to be:

$$\mathcal{M}^d = \bigcup M_n(\mathbb{C})^d,$$

where $M_n(\mathbb{C})^d$ is d-tuples of n by n matrices.
We define the **real matrix universe** to be:

$$\mathcal{S}^d = \bigcup S_n(\mathbb{C})^d,$$

where $S_n(\mathbb{C})^d$ is the set of d-tuples of n by n Hermitian matrices.
The Löwner ordering now holds entrywise- that is,

$$A \leq B \text{ if } A_i \leq B_i \qquad i = 1, \ldots, d$$

Noncommutative sets

Free set $D \subseteq \mathcal{M}^d$:

1. $X, Y \in D \Rightarrow X \oplus Y \in D$
2. $X \in D$ implies $U^* X U \in D$ whenever U is unitary.

We say D is **(open, convex)** whenever each $D \cap M_n(\mathbb{C})^d$ is (open, convex.)

Noncommutative functions

Let D be a free set. We define a **(real) noncommutative function** $f : D \to \mathcal{M}$ to satisfy

1. f is graded,
2. $f(X \oplus Y) = f(X) \oplus f(Y)$,
3. $f(U^* X U) = U^* f(X) U$ whenever U is unitary.

free Pick functions

Define the upper half plane Π^d to be the set of tuples with

$$\Pi^d = \{X \in \mathcal{M} : \frac{X - X^*}{2i} > 0\}.$$

A free Pick function is a function f with

$$f : \Pi^d \to \overline{\Pi}.$$

nc matrix functions

A free function f is matrix monotone if

$$A \leq B \Rightarrow f(A) \leq f(B).$$

A free function f is matrix convex if

$$f(\frac{A+B}{2}) \leq \frac{f(A)+f(B)}{2}.$$

Löwner's Theorem

Theorem (Pascoe-TD 2017, Palfia 2020)

Let D be a convex domain in $\mathcal{S}^d$. A free function $f : D \to \mathcal{S}$ is matrix monotone if and only if f analytically continues to Π^d as a function in the free Pick class.

Theorem (Pascoe 2018)

Let R_1 and R_2 be closed real operator systems. Let $D \subset \mathcal{S}(R_1)$ be a free domain with each D_n convex and open as a subset of $S_n(\mathbb{C}) \otimes R_1$. A function $f : D \to \mathcal{S}(R_2)$ is matrix monotone if and only if f extends to a continuous free function $F : \Pi(R_1) \cup D \to \overline{\Pi(R_2)}$.

Löwner's Theorem

Our first go at a proof of Löwner's theorem runs much along the same lines as classical approaches, involving

- establishing a Hankel-type condition related in spirit to the Hamburger moment problem,
- and constructing a model that automatically analytically continues to the "real" part of the domain (self-adjoint tuples).

free Nevanlinna representation

This came with an nc version of the Nevanlinna representation.

Theorem (Pascoe, TD 2017)

There exist a Hilbert space $\mathcal{H}$, a self-adjoint A, a positive decomposition Y, and vector V so that for all $Z \in \Pi^d$

$$f(Z) = (v^* \otimes I)(A \otimes I - \delta_Y(Z))^{-1}(v \otimes I)$$

if and only if f is a free Pick function and

$$\liminf_{s \to \infty} s \, |f(is_\chi)| < \infty.$$

Butterfly realizations

Helton, McCullough, and Vinnikov developed a Kraus-flavored realization for rational matrix convex functions in 2006.

Theorem (Helton, McCullough, Vinnikov '06)

Let $r : G \subset \mathcal{S}^d \to \mathcal{S}$ denote a noncommutative rational function on a domain G containing 0. If r is matrix convex near 0, then r has a realization of the form

$$r(X) = r_0 + L(X) + \Lambda(X)^*(1 - \Gamma(X))^{-1}\Lambda(X)$$

for a scalar r_0, a real linear function L, Λ affine linear, and $\Gamma(X) = \sum A_i \otimes X_i$ for self-adjoint matrices A_i.

$$f(x) = a + bx + \int_{[-1,1]} \frac{x^2}{1+tx}\, d\mu(t)$$

What is the royal road?

Euclid told Ptolemy “there is no royal road to geometry”.

But it turns out there is a royal road to Löwner’s theorem in several variables - namely, Löwner’s theorem in one variable.

The royal road is a method of applying classical one variable theorems in the several variable (nc free) setting.

Royal road approach

The royal road approach essentially axiomatizes the central properties of functions that make the automatic analyticity of classical one-variable structure theorems work -

- Structured domains
- Local boundedness
- local domination by derivatives
- Amenability to approximation - closure under convolution, e.g.

At the heart of the method is Pascoe's surprising and powerful continuation theorem, the *wedge-of-the-edge* theorem (see Pascoe BLMS and Pascoe CMB). Wedge-of-the-edge results allow the stitching together of slice data into domains of analyticity.

A royal road: dominions

A **dominion** is a set of domains, $\mathcal{G}$, in $\mathbb{R}^d$ satisfying:

- **Translation invariance**: For all $G \in \mathcal{G}$ and $r \in \mathbb{R}^d$, $G + r \in \mathcal{G}$.
- **Scale invariance:** If $t > 0$ and $G \in \mathcal{G}$, $tG \in \mathcal{G}$.
- **Closure under intersection:** For all $G, H \in \mathcal{G}$, $G \cap H \in \mathcal{G}$.
- **Locality:** For any $x \in \mathbb{R}^d$ and $\varepsilon > 0$, $B_{\mathbb{R}}(x, \varepsilon) \in \mathcal{G}$.

The class of convex sets in $\mathbb{R}^d$ is an example of a dominion, as is the class of all open sets in $\mathbb{R}^d$.

A royal road: sovereign classes

A **sovereign class** $\mathcal{F}$ is a set of functions on domains contained in a dominion $\mathcal{G}$ satisfying:

- **Functions:** For all $G \in \mathcal{G}$, $\mathcal{F}(G)$ is a set of locally bounded measurable functions.
- **Closure under localization:** If $f \in \mathcal{F}(G)$ and $H \subseteq G$ then $f|_H \in \mathcal{F}(H)$.
- **Closure under convolution:** The set of functions $\mathcal{F}(G)$ is convex and closed under pointwise weak limits.
- **Local boundedness:** Each $f \in \mathcal{F}$ is locally bounded and measurable on finite dimensional affine subspaces on each level.
- **One variable knowledge:** If $a_i \leq b_i$ for each i, then $f_{\overline{ab}}(t) := f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)$ analytically continues to $\mathbb{D}$ as a function of t.
- **Control** There is a map γ taking each pair (x, f) to a non-negative number satisfying
  1. For each $\varepsilon > 0$ there is a universal constant $c(\varepsilon)$ such that $\inf_{X \in B_{\mathbb{R}}(x,\varepsilon)} \gamma(x, f) \leq c(\varepsilon) \|f\|_{B_{\mathbb{R}}(x,\varepsilon)}$.
  2. There is a universal positive valued function e on $\mathbb{R}^+$ satisfying the following. Write $f_{\overline{ab}}(t) = \sum a_n t^n$. Then, $\|a_n\| \leq \gamma(x, f) e(\|b - a\|)$.
  3. If $H \subseteq G$ and $x \in H$ then $\gamma(x, f|_H) \leq \gamma(x, f)$.

Royal road theorem

Functions that meet the set of axioms are said to belong to a sovereign class. The main result of the royal road method asserts that these are precisely the axioms that lead to automatic analyticity.

Theorem (Pascoe-T.D. 2019)

Every function in a sovereign class is real analytic.

Proof idea

The basic idea of the proof is that functions in sovereign classes can be worked with locally, where they are analytic. One-variable parametrization gives local analytic continuations, (which allows application of one-variable theorems) and then resulting one variable regularity can be stitched together into global real analyticity by way of the wedge of the edge theorem on each level, and then into the whole domain.

In Pascoe-T.D. 2020 (*Regal Path*), a simplified set of axioms can be used to extract a proof of the commutative Löwner theorem (Agler-McCarthy-Young 2012 and Pascoe 2018) from the noncommutative theorem (Pascoe-T.D. 2016, Palfia 2020, Pascoe 2019)

A road to realizations

Define the **positive-orthant norm of the n-th derivative at X** by

$$\|D^n f(X)\|_+ = \sup_{\|H\|=1, H>0, m} \|D^n f(X^{\oplus m})[H]\|.$$

If the n-th derivative does not exist in some positive direction, we formally set $\|D^n f(X)\|_+ = \infty$.

Theorem

Matrix monotone functions are a sovereign class.

Controlled by

$$\gamma(X, f) = \|f(X)\| + \|Df(X)\|_+ .$$

Theorem

Matrix convex functions are a sovereign class.

$$\gamma(X, f) = \|f(X)\| + \|Df(X)\|_+ + \left\|D^2 f(X)\right\|_+ .$$

A road to realizations

Theorem

Matrix monotone functions are a sovereign class.

Theorem

Matrix convex functions are a sovereign class.

That is, matrix monotone and matrix convex nc functions have (real) convergent power series. The representations will follow along similar lines to the classical theory - establish positivity of a Hankel-type construction, and use it to construct a representation (in the spirit of Hilbert spaces standing in for measures).

Convex functions

In one variable, convex functions are characterized by positivity of the Hessian matrix. The same characterization holds in the nc setting for the appropriate notion.

That is, for a matrix convex function f, we see that

$$D^2 f(X)[H] = \frac{d^2}{dt^2} f(X + tH)|_{t=0} \geq 0$$

For a survey, see e.g. Helton-Klep-McCullough '13 and related work.

Kraus theorem

(The construction that follows is related to Nevanlinna's approach to the Hamburger moment problem.)
Let $f(X) = \sum c_\alpha X^\alpha$ be matrix convex near 0.
The second derivative of f is of the form

$$D^2 f(X)[H] = 2 \sum_{\alpha,\beta,\gamma,i,j} c_{\alpha^* x_i \gamma x_j \beta} X^{\alpha^*} H_i X^\gamma H_j X^\beta \geq 0.$$

Evaluating the expression at $X = \left[\begin{smallmatrix} X & 0 \\ 0 & X \end{smallmatrix}\right]$ and $H = \left[\begin{smallmatrix} 0 & H \\ H & 0 \end{smallmatrix}\right]$ removes the γ. Eventually, we get that the Hankel-type matrix

$$[c_{\alpha^* x_i x_j \beta}]_{x_j \beta, x_i \alpha} \geq 0.$$

The rest of the argument follows from considering the related Hilbert space and power series manipulation.

The butterfly realization

The result is the following Kraus theorem (which extends the Helton-Klep-McCullough result and simplifies the argument).

Theorem (Pascoe-TD '19)

Let f be a locally bounded matrix convex function defined on some matrix convex set of self adjoints containing 0*. There are self-adjoint T_i, vector Q_i, a scalar a_0, and a linear function L such that*

$$f(X) = a_0 + L(X) + \left(\sum Q_i X_i\right)^* \left(I - \sum T_i X_i\right)^{-1} \left(\sum Q_i X_i\right).$$

Further considerations

The Löwner-Nevalinna realization for a function $f(X) = c_\alpha X^\alpha$ turns out to pivot on the positivity of a Hankel-type matrix (as in the classical case).

Likewise, for Kraus functions, we consider the positivity of the Hankel-type matrix

$$[c_{\alpha^* x_i x_j \beta}]_{x_j \beta, x_i \alpha}$$

Are there others?

Further considerations

Matrix monotone: controlled by

$$\gamma(X, f) = \|f(X)\| + \|Df(X)\|_+ .$$

Matrix convex: controlled by

$$\gamma(X, f) = \|f(X)\| + \|Df(X)\|_+ + \left\|D^2 f(X)\right\|_+ .$$

Is there a family of "jerk" functions (third derivative?)

Further considerations

The royal road approach also points the way to a deep theorem of Pascoe - the free universal monodromy theorem. The existence of powerful analytic continuations is perhaps less surprising given the result.

Theorem (Pascoe)

A connected open free set satisfies the monodromy theorem.

The consequences of the free universal monodromy theorem are striking (see Pascoe's talk tomorrow, perhaps).

Thank you!