Wold decompositions for representations of C^*-algebras associated with noncommutative varieties

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Wold decomposition, 1954

- S is the unilateral shift defined on the Hardy space $H^2(\mathbb{D})$ by $(S\varphi)(z) := zf(z)$.

Theorem

If $V \in B(\mathcal{K})$ is an isometry, then $\mathcal{K}$ admits an orthogonal decomposition $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$, where $\mathcal{K}_0$ and $\mathcal{K}_1$ are reducing subspaces for V such that

(i) *$V|_{\mathcal{K}_0}$ is unitarily equivalent to a direct sum of copies of the unilateral shift S,*

(ii) *$V|_{\mathcal{K}_1}$ is a unitary operator.*

Moreover, the decomposition is uniquely determined :
$\mathcal{K}_1 = \bigcap_{n=0}^{\infty} V^n\mathcal{K}$, $\quad \mathcal{K}_0 = \oplus_{n=0}^{\infty} V^n\mathcal{L}$, $\quad$ *where* $\mathcal{L} = \mathcal{K} \ominus V\mathcal{K}$.

Multivariable noncommutative case

- Let H_n be a complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$. The full Fock space of H_n defined by

$$F^2(H_n) := \bigoplus_{k\geq 0} H_n^{\otimes k},$$

  where $H_n^{\otimes 0} := \mathbb{C}1$.
- The left creation operators $S_i : F^2(H_n) \to F^2(H_n)$ are defined by

$$S_i\varphi := e_i \otimes \varphi, \qquad \varphi \in F^2(H_n).$$

- $(S_1, \ldots, S_n)$ plays the role of universal model for row contractions :

$$\{(T_1, \ldots, T_n) \in B(\mathcal{H})^n : \ T_1T_1^* + \cdots + T_nT_n^* \leq I\}.$$

Wold decomposition

- Popescu, Trans. AMS, 1989

Theorem

Let $V_1, \ldots, V_n$ be isometries on a Hilbert space $\mathcal{K}$ such that $V_1 V_1^* + \cdots + V_n V_n^* \leq I$. Then $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$, where $\mathcal{K}_0$ and $\mathcal{K}_1$ are reducing subspaces for $V_1, \ldots, V_n$ such that

(i) $[V_1|_{\mathcal{K}_0} \cdots V_n|_{\mathcal{K}_0}]$ *is a pure row isometry, i.e unitarily equivalent to* $[S_1 \otimes I_{\mathcal{L}} \cdots S_n \otimes I_{\mathcal{L}}]$,

(ii) $[V_1|_{\mathcal{K}_1} \cdots V_n|_{\mathcal{K}_1}]$ *is a Cuntz row isometry, i.e*

$$V_1 V_1^*|_{\mathcal{K}_1} + \cdots + V_n V_n^*|_{\mathcal{K}_1} = I_{\mathcal{K}_1}.$$

The decomposition is uniquely determined.

- Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, \ldots, g_n$ and the identity g_0.
- If $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$ and $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$, we denote $X_\alpha := X_{i_1} \cdots X_{i_k}$ and $X_{g_0} := I_{\mathcal{H}}$.
- Let $Z_1, \ldots, Z_n$ be noncommutative indeterminates. A formal power series $f := \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$, $a_\alpha \in \mathbb{C}$, is called free holomorphic function on the noncommutative ball

$$[B(\mathcal{H})^n]_1 = \{(X_1, \ldots, X_n) \in B(\mathcal{H})^n : \|X_1X_1^* + \cdots + X_nX_n^*\| < 1\},$$

if the series $\sum_{k=0}^\infty \sum_{|\alpha|=k} a_\alpha X_\alpha$ is convergent in the operator norm topology for any $(X_1, \ldots, X_n) \in [B(\mathcal{H})^n]_1$, and any $\mathcal{H}$.

- f is called positive regular free holomorphic function if $a_\alpha \geq 0$ for any $\alpha \in \mathbb{F}_n^+$, $a_{g_0} = 0$, and $a_{g_i} > 0$ if $i = 1, \ldots, n$.

Noncommutative domains

- We define the noncommutative regular domain $\mathcal{D}_f^m(\mathcal{H})$, $m = 1, 2, \ldots$, to be the set of all $X := (X_1, \ldots, X_n) \in B(\mathcal{H})^n$ such that

$$\Phi_{f,X}(I) \leq I \quad \text{and} \quad (id - \Phi_{f,X})^m(I) \geq 0$$

where $\Phi_{f,X} : B(\mathcal{H}) \to B(\mathcal{H})$ is defined by

$$\Phi_{f,X}(Y) := \sum_{|\alpha| \geq 1} a_\alpha X_\alpha Y X_\alpha^*, \quad Y \in B(\mathcal{H}),$$

and the convergence is in the weak operator topology.

- Define $b_{g_0} := 1$ and

$$b_\alpha := \sum_{j=1}^{|\alpha|} \sum_{\substack{\gamma_1 \cdots \gamma_j = \alpha \\ |\gamma_1| \geq 1, \ldots, |\gamma_j| \geq 1}} a_{\gamma_1} \cdots a_{\gamma_j} \binom{j+m-1}{m-1} \qquad \text{if } |\alpha| \geq 1.$$

Universal model

- Let $D_i : F^2(H_n) \to F^2(H_n)$, $i \in \{1, \ldots, n\}$, be the diagonal operators defined by setting

$$D_i e_\alpha := \sqrt{\frac{b_\alpha}{b_{g_i\alpha}}} e_\alpha, \qquad \alpha \in \mathbb{F}_n^+,$$

  where $\{e_\alpha\}_{\alpha\in\mathbb{F}_n^+}$ is the orthonormal basis of $F^2(H_n)$.

- The n-tuple $(W_1, \ldots, W_n)$ of weighted shifts,

$$W_i := S_i D_i,$$

  associated with the noncommutative domain $\mathcal{D}_f^m$, plays the role of universal model for the pure elements of $\mathcal{D}_f^m$ (Popescu, Mem. AMS, 2010, JFA, 2008).

Noncommutative varieties $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$

- Let $\mathcal{Q} \subset \mathbb{C}\langle Z_1, \ldots, Z_n \rangle$ be a fixed set of noncommutative polynomials such that $q(0) = 0$ for any $q \in \mathcal{Q}$
- Define the noncommutative variety $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$ to be the set

$$\{(X_1, \ldots, X_n) \in \mathcal{D}_f^m(\mathcal{H}) : \; q(X_1, \ldots, X_n) = 0 \quad \text{for any} \quad q \in \mathcal{Q}\}$$

  and assume that $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H}) \neq \{0\}$.
- The universal model $(B_1, \ldots, B_n)$ associated with $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{H})$ is given by

$$B_i^* = W_i^*|_{\mathcal{N}_{f,\mathcal{Q}}}, \quad 1 = 1, \ldots, n,$$

  acting on a model space $\mathcal{N}_{f,\mathcal{Q}} \subset F^2(H_n)$ which is a joint invariant subspace under the adjoints $W_1^*, \ldots, W_n^*$.

Single variable case : $n = 1$ and $\mathcal{Q} = 0$

- If $m = 1$ and $f = Z$, the corresponding domain $\mathcal{D}_f^m(\mathcal{H})$ coincides with

$$[B(\mathcal{H})]_1 := \{X \in B(\mathcal{H}) : \ \|X\| \leq 1\}.$$

In this case, the universal model is the unilateral shift S acting on the Hardy space $H^2(\mathbb{D})$.

- If $m \geq 2$ and $f = Z$, the corresponding domain coincides with the set of all m-hypercontractions studied by Agler, Olofsson, Ball-Bolotnikov. The corresponding universal model is the unilateral shift acting on the weighted Bergman space, which is a reproducing kernel Hilbert space corresponding to the kernel $k_m(z, w) = \frac{1}{(1-z\bar{w})^m}$, $z, w \in \mathbb{D}$.

Multivariable commutative case : $n \geq 2$

Case : $\mathcal{Q} := \{Z_i Z_j - Z_j Z_i, i, j = 1, \ldots, n\}$

- If $m \geq 1$ and $f = Z_1 + \cdots + Z_n$ the corresponding commutative variety was studied by Drury, Arveson, Bhattacharyya-Eschmeier-Sarkar, Popescu (when $m = 1$), Athavale, Müller, Müler-Vasilescu, and Curto-Vasilescu (when $m \geq 2$).
  The corresponding universal model is the n-tuple $(M_{z_1}, \ldots, M_{z_n})$ of multipliers by the coordinate functions, acting on the reproducing kernel Hilbert space corresponding to the kernel

$$k_m(\mathbf{z}, \mathbf{w}) = \frac{1}{(1 - z_1 \bar{w}_1 - \cdots - z_n \bar{w}_n)^m}, \qquad \mathbf{z}, \mathbf{w} \in \mathbb{B}_n,$$

on the unit ball of $\mathbb{C}^n$.

Multivariable commutative case : $n \geq 2$

Case : $\mathcal{Q} := \{Z_i Z_j - Z_j Z_i, i, j = 1, \ldots, n\}$

- When $m \geq 1$ and f is a positive regular commutative polynomial, the commutative variety $\mathcal{V}_{f,\mathcal{Q}}$ was studied by S. Pott . In this case, the universal model $(M_{z_1}, \ldots, M_{z_n})$ acts on a reproducing kernel Hilbert space of holomorphic functions on a Reinhardt domain in $\mathbb{C}^n$ uniquely determined by f.

Multivariable noncommutative case

Case : $n \geq 2$ and $\mathcal{Q} = 0$

- When $m = 1$, $f = Z_1 + \cdots + Z_n$, the noncommutative domain $\mathcal{D}_f^m(\mathcal{H})$ coincides with the closed unit ball $[B(\mathcal{H})^n]_1$, the study of which has generated a free analogue of Sz.-Nagy-Foiaş theory. The corresponding universal model is the n-tuple of left creation operators $(S_1, \ldots, S_n)$.

- When $m \geq 1$, $n \geq 1$, and f is any positive regular free holomorphic function the domain $\mathcal{D}_f^m$ was studied by Popescu (Mem. AMS, 2010 and JFA 2008). In this case, the corresponding universal model is the n-tuple of weighted left creation operators $(W_1, \ldots, W_n)$ acting on the full Fock space with n generators.

Multivariable noncommutative case

Case : $n \geq 2$, $m \in \mathbb{N}$, and $\mathcal{Q} \subset \mathbb{C}\langle Z_1, \ldots, Z_n \rangle$

- The study of general noncommutative varieties $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ in $B(\mathcal{H})^n$, where $m \geq 1$, f is a positive regular free holomorphic function, and $\mathcal{Q} \subset \mathbb{C}\langle Z_1, \ldots, Z_n \rangle$ is any set of noncommutative polynomials such that $q(0) = 0$ for any $q \in \mathcal{Q}$, was initiated in 2006 ($m = 1$, $f = Z_1 + \cdots + Z_n$).

• G. Popescu, Operator theory on noncommutative varieties, *Indiana Univ. Math. J.*, 2006.

• G. Popescu, Noncommutative Berezin transforms and multivariable operator model theory, *J. Funct. Anal.*, 2008.

• G. Popescu, Operator theory on noncommutative domains, *Mem. Amer. Math. Soc.*, 2010 (Case $m = 1$).

Noncommutative Berezin kernel

- Let $T := (T_1, \dots, T_n) \in \mathcal{D}_f^m(\mathcal{H})$. The noncommutative Berezin kernel is the map $K_{f,T} : \mathcal{H} \to F^2(H_n) \otimes \overline{\Delta_{f,T}(\mathcal{H})}$ defined by

$$K_{f,T}h := \sum_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha} e_\alpha \otimes \Delta_{f,T} T_\alpha^* h, \qquad h \in \mathcal{H},$$

where $\Delta_{f,T} := \left[(I - \Phi_{f,T})^m(I)\right]^{1/2}$ and

$$\Phi_{f,T}(Y) = \sum_{|\alpha| \geq 1} a_\alpha T_\alpha Y T_\alpha^*, \quad Y \in B(\mathcal{H})$$

Wold decomposition

Definition

We say that $X = (X_1, \ldots, X_n)$ is a *pure tuple* in $\mathcal{D}_f^m(\mathcal{H})$ if

$$\text{SOT-}\lim_{s\to\infty} \Phi_{f,X}^s(I_{\mathcal{H}}) = 0.$$

If X satisfies the relation

$$\Phi_{f,X}(I_{\mathcal{H}}) = I_{\mathcal{H}},$$

we say that X is a *Cuntz tuple* in the domain $\mathcal{D}_f^m(\mathcal{H})$.

- Let $\mathbf{W} := (W_1, \ldots, W_n)$ be the universal model of $\mathcal{D}_f^m$. We denote by $C^*(\mathbf{W})$ the C^*-algebra generated by $W_1, \ldots, W_n$ and the identity.

Wold decomposition

Theorem

Let π be a unital $*$-representation of the C^*-algebra $C^*(\mathbf{W})$ on a separable Hilbert space $\mathcal{K}$ and set $V_i := \pi(W_i)$. Then the noncommutative Berezin kernel $K_{f,V}$ is a partial isometry. Setting

$$\mathcal{K}^{(0)} := \operatorname{range} K_{f,V}^* \quad \text{and} \quad \mathcal{K}^{(1)} := \ker K_{f,V},$$

the orthogonal decomposition $\mathcal{K} = \mathcal{K}^{(0)} \bigoplus \mathcal{K}^{(1)}$ has the properties.

(i) *$\mathcal{K}^{(0)}$ and $\mathcal{K}^{(1)}$ are reducing subspaces for each operator V_i.*

(ii) *$V|_{\mathcal{K}^{(0)}} := (V_1|_{\mathcal{K}^{(0)}}, \ldots, V_n|_{\mathcal{K}^{(0)}})$ is a pure tuple in $\mathcal{D}_f^m(\mathcal{K}^{(0)})$.*

(iii) *$V|_{\mathcal{K}^{(1)}} := (V_1|_{\mathcal{K}^{(1)}}, \ldots, V_n|_{\mathcal{K}^{(1)}})$ is a Cuntz tuple in $\mathcal{D}_f^m(\mathcal{K}^{(1)})$.*

Moreover, $K_{f,V}|_{\mathcal{K}^{(0)}}$ is a unitary operator satisfying relation

$$V_i|_{\mathcal{K}^{(0)}} = \left(K_{f,V}|_{\mathcal{K}^{(0)}}\right)^* (W_i \otimes I_{\mathcal{D}}) \left(K_{f,V}|_{\mathcal{K}^{(0)}}\right),$$

where $\mathcal{D} := \operatorname{range} \Delta_{f,V}(I_{\mathcal{K}})$.
In addition, the orthogonal decomposition of $\mathcal{K}$ is uniquely determined by the properties (i), (ii), and (iii) and we have

$$\mathcal{K}^{(0)} = \bigoplus_{\alpha \in \mathbb{F}_n^+} V_\alpha(\mathcal{D}), \qquad \text{where} \quad \mathcal{D} = \mathcal{K} \ominus \left(\bigoplus_{i=1}^n \overline{V_i \mathcal{K}} \right),$$

and

$$\mathcal{K}^{(1)} = \bigcap_{s=0}^{\infty} \left(\bigoplus_{\alpha \in \mathbb{F}_n^+, |\alpha| = s} \overline{V_\alpha(\mathcal{K})} \right).$$

Constrained Berezin kernel

Definition

The *constrained noncommutative Berezin kernel* associated with the n-tuple $T \in \mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{H})$ is the bounded operator $K_{f,T,\mathcal{Q}} : \mathcal{H} \to \mathcal{N}_{f,\mathcal{Q}} \otimes \overline{\Delta_{f,T}(\mathcal{H})}$ defined by

$$K_{f,T,\mathcal{Q}} := (P_{\mathcal{N}_{f,\mathcal{Q}}} \otimes I_{\overline{\Delta_{f,T}(\mathcal{H})}})K_{f,T},$$

where $K_{f,T}$ is the Berezin kernel associated with $T \in \mathcal{D}^m_f(\mathcal{H})$ and $\mathcal{N}_{f,\mathcal{Q}} \subset F^2(H_n)$ is the model space on which the universal model $(B_1, \ldots, B_n)$ is acting.

Constrained Berezin kernel

Properties :

- $K_{f,T,\mathcal{Q}} T_i^* = (B_i^* \otimes I) K_{f,T,\mathcal{Q}}, \qquad i \in \{1, \ldots, n\}.$
- When T is a pure n-tuple, i.e. $\Phi_{f,T}^k(I) \to 0$, as $k \to \infty$, the constrained noncommutative Berezin kernel $K_{f,T,\mathcal{Q}}$ is an isometry.
- Assume that $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha Z_\alpha$ is a positive regular free holomorphic function such that $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha W_\alpha W_\alpha^*$ is convergent in the operator norm.

Wold decomposition : C^*-version

Theorem

Let $\mathbf{B} := (B_1, \ldots, B_n)$ *be the universal model associated with the noncommutative variety* $\mathcal{V}^m_{f,\mathcal{Q}}$ *and let* $\pi : C^*(\mathbf{B}) \to B(\mathcal{K})$ *be a unital* $*$*-representation of* $C^*(\mathbf{B})$ *on a separable Hilbert space* $\mathcal{K}$. *Then* π *decomposes into a direct sum*

$$\pi = \pi_0 \oplus \pi_1 \ \text{ on } \ \mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1,$$

where π_0 *and* π_1 *are disjoint representations of* $C^*(\mathbf{B})$ *on the Hilbert spaces*

$$\mathcal{K}_0 := \overline{\text{span}} \left\{ \pi(B_\beta)[\left(id - \Phi_{f,\pi(\mathbf{B})}\right)^m (I_\mathcal{K})]\mathcal{K} : \ \beta \in \mathbb{F}_n^+ \right\} \quad \text{and}$$

$$\mathcal{K}_1 := \mathcal{K} \ominus \mathcal{K}_0, \ \text{where } \pi(\mathbf{B}) := (\pi(B_1), \ldots, \pi(B_n)).$$

Wold decomposition : C^*-version

Moreover, up to an isomorphism,

$$\mathcal{K}_0 \simeq \mathcal{N}_{f,\mathcal{Q}} \otimes \mathcal{G}, \quad \pi_0(X) = X \otimes I_{\mathcal{G}} \quad \text{for } \ X \in C^*(\mathbf{B}),$$

where $\mathcal{G}$ is a Hilbert space with

$$\dim \mathcal{G} = \dim \left[\text{range } \left(id - \Phi_{f,\pi(\mathbf{B})}\right)^m (I_{\mathcal{K}})\right],$$

and π_1 is a $*$-representation which annihilates the compact operators in $C^*(B_1, \ldots, B_n)$ and $\Phi_{f,\pi_1(\mathbf{B})}(I_{\mathcal{K}_1}) = I_{\mathcal{K}_1}$.

- When $f = Z_1 + \cdots + Z_n$, $m = 1$, and $\mathcal{Q} = \{Z_iZ_j - Z_jZ_i\}$, then **B** is the universal model on the Drury-Arveson space. The theorem was obtained by Arveson, Acta Math., 1998.

Wold decomposition : geometric version

Theorem

Let π be a unital $*$-representation of the C^*-algebra $C^*(\mathbf{B})$ on a separable Hilbert space $\mathcal{K}$ and set $V_i := \pi(B_i)$. Then the noncommutative Berezin kernel $K_{f,V,\mathcal{Q}}$ associated with the noncommutative variety $\mathcal{V}_{f,\mathcal{Q}}^m$ is a partial isometry. Setting

$$\mathcal{K}^{(0)} := \operatorname{range} K_{f,V,\mathcal{Q}}^* \quad \text{and} \quad \mathcal{K}^{(1)} := \ker K_{f,V.\mathcal{Q}},$$

the orthogonal decomposition $\mathcal{K} = \mathcal{K}^{(0)} \oplus \mathcal{K}^{(1)}$ has the following properties.

(i) $\mathcal{K}^{(0)}$ *and* $\mathcal{K}^{(1)}$ *are reducing subspaces for each operator* V_i.

(ii) $V|_{\mathcal{K}^{(0)}} := (V_1|_{\mathcal{K}^{(0)}}, \ldots, V_n|_{\mathcal{K}^{(0)}})$ *is a pure tuple in* $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{K}^{(0)})$.

(iii) $V|_{\mathcal{K}^{(1)}} := (V_1|_{\mathcal{K}^{(1)}}, \ldots, V_n|_{\mathcal{K}^{(1)}})$ *is a Cuntz tuple in* $\mathcal{V}_{f,\mathcal{Q}}^m(\mathcal{K}^{(1)})$.

Wold decomposition : geometric version

Moreover, the constrained Berezin kernel $K_{f,V,\mathcal{Q}}|_{\mathcal{K}^{(0)}}$ is a unitary operator satisfying relation

$$V_i|_{\mathcal{K}^{(0)}} = \left(K_{f,V,\mathcal{Q}}|_{\mathcal{K}^{(0)}}\right)^* (B_i \otimes I_{\mathcal{D}}) \left(K_{f,V,\mathcal{Q}}|_{\mathcal{K}^{(0)}}\right),$$

where $\mathcal{D} := \mathrm{range}\, \Delta_{f,V}(I_{\mathcal{K}})$ and

$$\Delta_{f,V}(I_{\mathcal{K}}) := [(id - \Phi_{f,V})^m (I_{\mathcal{K}})]^{1/2}$$

is an orthogonal projection. In addition, the space $\mathcal{K}^{(0)}$ admits the decomposition

$$\mathcal{K}^{(0)} = \mathcal{D} \bigoplus \overline{\mathrm{span}} \left\{ V_\alpha(\mathcal{D}) : \ \alpha \in \mathbb{F}_n^+, |\alpha| \geq 1 \right\}.$$

Wold decomposition : operator version

Theorem

Let π be a unital $*$-representation of the C^*-algebra $C^*(\mathbf{B})$ on a separable Hilbert space $\mathcal{K}$ and set $V_i := \pi(B_i)$. Then there are two subspaces $\mathcal{K}^{(0)}$ and $\mathcal{K}^{(1)}$ of $\mathcal{K}$ such that

(i) $I_\mathcal{K} = P_{\mathcal{K}^{(0)}} + P_{\mathcal{K}^{(1)}}$ *and* $P_{\mathcal{K}^{(0)}} P_{\mathcal{K}^{(1)}} = 0$, *where* $P_{\mathcal{K}^{(0)}}$ *and* $P_{\mathcal{K}^{(1)}}$ *are orthogonal projections.*

(ii) $P^{(0)}$ *and* $P^{(1)}$ *commute with each operator* V_i.

(iii) *The orthogonal projection* $P_{\mathcal{K}^{(0)}}$ *satisfies the relations*

$$P_{\mathcal{K}^{(0)}} = \lim_{q\to\infty} \sum_{s=0}^{q} \binom{s+m-1}{m-1} \left\{\Phi^s_{f,V}[\Delta_{f,V}(I_\mathcal{K})]^2)\right\}$$
$$= I_\mathcal{K} - \text{SOT-}\lim_{s\to\infty} \Phi^s_{f,V}(I_\mathcal{K})$$

Wold decomposition : operator version

and

$$\mathcal{K}^{(0)} = \left\{ \xi \in \mathcal{K} : \lim_{s\to\infty} \Phi^s_{f,V}(I_\mathcal{K})\xi = 0 \right\}.$$

(iv) The orthogonal projection $P_{\mathcal{K}^{(1)}}$ satisfies the relation

$$P_{\mathcal{K}^{(1)}} = \text{SOT-}\lim_{s\to\infty} \Phi^s_{f,V}(I_\mathcal{K})$$

and

$$\mathcal{K}^{(1)} = \{ \xi \in \mathcal{K} : \Phi^s_{f,V}(I_\mathcal{K})\xi = \xi \text{ for every } s \in \mathbb{N} \}$$

(v) SOT-$\lim_{s\to\infty} \Phi^s_{f,V}(I_\mathcal{K}) P_{\mathcal{K}^{(0)}} = 0$ and $\Phi^s_{f,V}(I_\mathcal{K}) P_{\mathcal{K}^{(1)}} = P_{\mathcal{K}^{(1)}}$ for any $s \in \mathbb{N}$.

Uniqueness of Wold decomposition

Theorem

Let $\pi : C^*(\mathbf{B}) \to B(\mathcal{K})$ be a unital $*$-representation of $C^*(\mathbf{B})$ on a separable Hilbert space $\mathcal{K}$ and let $V_i := \pi(B_i)$.

(i) *If $\mathcal{M}, \mathcal{N}$ are reducing subspaces under each operator V_i such that $(V_1|_{\mathcal{M}}, \ldots, V_n|_{\mathcal{M}})$ is a pure tuple in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{M})$ and $(V_1|_{\mathcal{N}}, \ldots, V_n|_{\mathcal{N}})$ is a Cuntz tuple in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{N})$, then $\mathcal{M} \perp \mathcal{N}$.*

(ii) *If $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$ (algebraically) and $\mathcal{M}, \mathcal{N}$ are as in part (i), then*

$$\mathcal{M} = \mathcal{K}^{(0)} \quad \text{and} \quad \mathcal{N} = \mathcal{K}^{(1)}.$$

(iii) *A subspace $\mathcal{M} \subset \mathcal{K}$ is reducing under each operator V_i if and only if $\mathcal{M} = \mathcal{M}^{(0)} \oplus \mathcal{M}^{(1)}$, where $\mathcal{M}^{(0)} \subset \mathcal{K}^{(0)}$ and $\mathcal{M}^{(1)} \subset \mathcal{K}^{(1)}$ are reducing subspaces under each V_i.*

Representations of C^*-algebras

Theorem

Let $\pi : C^*(\mathbf{B}) \to B(\mathcal{K})$ be a unital $*$-representation on a separable Hilbert space $\mathcal{K}$ and let $V_i := \pi(B_i)$. Then the following statements are equivalent.

(i) $(V_1, \ldots, V_n)$ *is a pure n-tuple in* $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{K})$.

(ii) $(V_1, \ldots, V_n)$ *is unitarily equivalent to* $(B_1 \otimes I_{\mathcal{G}}, \ldots, B_n \otimes I_{\mathcal{G}})$ *for some Hilbert space* $\mathcal{G}$.

(iii) *The only reducing subspace* $\mathcal{M} \subset \mathcal{K}$ *for every* V_i *such that* $(V_1|_{\mathcal{M}}, \ldots, V_n|_{\mathcal{M}})$ *is a Cuntz tuple in* $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{K})$ *is* $\mathcal{M} = \{0\}$.

(iv) $K_{f,V,\mathcal{Q}}$ *is a unitary operator,*

$$V_i = K^*_{f,V,\mathcal{Q}}(B_i \otimes I_{\mathcal{D}})K_{f,V,\mathcal{Q}}, \quad \mathcal{D} := \operatorname{range} \Delta_{f,V}(I_{\mathcal{K}}).$$

Representations of C^*-algebras

- A subspace $\mathcal{E}$ of $\mathcal{H}$ is called *cyclic* for $T := (T_1, \ldots, T_n)$ if $\mathcal{H} = \overline{\text{span}}\{T_\alpha \mathcal{E} : \alpha \in \mathbb{F}_n^+\}$. The *multiplicity* of T is the minimum dimension of a cyclic subspace of T.

Theorem

Let $\pi : C^*(\mathbf{B}) \to B(\mathcal{K})$ be a pure, unital $*$-representation on a separable Hilbert space $\mathcal{K}$ and let $V_i := \pi(B_i)$. Then the following statements hold.

(i) *The multiplicity of $V := (V_1, \ldots, V_n)$ is equal to* $\dim \Delta_{f,V}(I_\mathcal{K})\mathcal{K}$.

(ii) *If $\sigma : C^*(\mathbf{B}) \to B(\mathcal{K}')$ is another pure, unital $*$-representation, then π and σ are unitarily equivalent if and only if they have the same multiplicity.*

Representations of C^*-algebras

(iii) The reducing subspaces $\mathcal{M} \subset \mathcal{K}$ of $V_1, \ldots, V_n$ are precisely of the form

$$\mathcal{M} = K_{f,V,\mathcal{Q}}(\mathcal{N}_{f,\mathcal{Q}} \otimes \mathcal{D}_0),$$

where $\mathcal{D}_0$ is a subspace of $\mathcal{D}$.

(iv) The joint invariant subspaces $\mathcal{M} \subset \mathcal{K}$ under $V_1, \ldots, V_n$ are precisely of the form

$$\mathcal{M} = K_{f,V,\mathcal{Q}}(\mathcal{G}),$$

where $\mathcal{G} \subset \mathcal{N}_{f,\mathcal{Q}} \otimes \mathcal{D}$ is a joint invariant subspace under $B_1 \otimes I_{\mathcal{D}}, \ldots B_n \otimes I_{\mathcal{D}}$.

Representations of C^*-algebras

- A Beurling type characterization of the joint invariant subspaces under $(B_1 \otimes I_{\mathcal{D}}, \ldots B_n \otimes I_{\mathcal{D}})$ was obtained by Popescu, Math. Ann., 2018.

Theorem

Let $\pi : C^*(\mathbf{B}) \to B(\mathcal{K})$ be a unital $*$-representation on a separable Hilbert space $\mathcal{K}$ and let $V_i := \pi(B_i)$. Then the following statements are equivalent.

(i) *$(V_1, \ldots, V_n)$ is a Cuntz n-tuple in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{M})$.*

(ii) *There is no nonzero $\xi \in \mathcal{K}$ such that* $\lim_{k\to\infty} \left\langle \Phi^k_{f,V}(I_{\mathcal{K}})\xi, \xi \right\rangle = 0$.

(iii) *The only reducing subspace $\mathcal{M} \subset \mathcal{K}$ for every operator V_i such that $(V_1|_{\mathcal{M}}, \ldots, V_n|_{\mathcal{M}})$ is a pure tuple in $\mathcal{V}^m_{f,\mathcal{Q}}(\mathcal{M})$ is $\mathcal{M} = \{0\}$.*

Representations of C^*-algebras

Theorem

Let $\pi : C^*(\mathbf{B}) \to B(\mathcal{K})$ be a Cuntz type unital $*$-representation on a separable Hilbert space $\mathcal{K}$ and let $V_i := \pi(B_i)$. Then

(i) *$\mathcal{M} \subset \mathcal{K}$ is an invariant subspace under each V_i if and only if*

$$\Phi_{f,V}(P_{\mathcal{M}}) \leq P_{\mathcal{M}};$$

(ii) *$\mathcal{M} \subset \mathcal{K}$ is a reducing subspace under each V_i if and only if*

$$\Phi_{f,V}(P_{\mathcal{M}}) = P_{\mathcal{M}}.$$

Irreducible representations

Theorem

Let $\pi : C^*(\mathbf{B}) \to B(\mathcal{K})$ *be a unital* $*$*-representation on a separable Hilbert space* $\mathcal{K}$ *and let* $V_i := \pi(B_i)$*. Then* π *is irreducible if and only if one of the following conditions is satisfied :*

(i) $\text{SOT-}\lim_{s\to\infty} \Phi^s_{f,V}(I_\mathcal{K}) = 0$ *and* $\dim \Delta_{f,V}(I_\mathcal{K})\mathcal{K} = 1$ *;*

(ii) $\Phi_{f,V}(I_\mathcal{K}) = I_\mathcal{K}$ *and there is no nontrivial subspace* $\mathcal{M} \subset \mathcal{K}$ *such that* $\Phi_{f,V}(P_\mathcal{M}) = P_\mathcal{M}$*.*

Classification

Theorem

Let π and π' be two unital $*$-representations of $C^*(\mathbf{B})$ on separable Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$, respectively, and let $\mathcal{K} = \mathcal{K}^{(0)} \oplus \mathcal{K}^{(1)}$ and $\mathcal{K} = \mathcal{K}'^{(0)} \oplus \mathcal{K}'^{(1)}$ be the corresponding Wold decompositions. Set $V_i := \pi(B_i)$ and $V_i := \pi'(B_i)$. Then π is unitarily equivalent to π' if and only if the following conditions are satisfied :

(i) $\dim \mathcal{D} = \dim \mathcal{D}'$, *where* $\mathcal{D} := \Delta_{f,V}(I_{\mathcal{K}})\mathcal{K}$ *and* $\mathcal{D} := \Delta_{f,V'}(I_{\mathcal{K}'})\mathcal{K}'$ *;*

(ii) *the n-tuples* $V|_{\mathcal{K}^{(1)}}$ *and* $V'|_{\mathcal{K}'^{(1)}}$ *are unitarily equivalent.*

Exact sequences of C^*-algebras

Definition

The algebra $\mathcal{O}(\mathcal{V}^m_{f,\mathcal{Q}})$ is the universal C^*-algebra generated by $\pi(B_1),\dots,\pi(B_n)$ and the identity, where $\mathbf{B}=(B_1,\dots,B_n)$ is the universal model of a noncommutative variety $\mathcal{V}^m_{f,\mathcal{Q}}$ and π is a *Cuntz type* $*$-representation of the C^*-algebra $C^*(\mathbf{B})$, i.e. $\Phi_{f,\pi(\mathbf{B})}(I)=I$, where $\pi(\mathbf{B}):=(\pi(B_1),\dots,\pi(B_n))$.

- If $n=1$, $f=Z$, $m=1$ and $\mathcal{Q}=0$, then $\mathcal{O}(\mathcal{V}^m_{f,\mathcal{Q}})=C(\mathbb{T})$.
- If $f=Z_1+\cdots+Z_n$, $n\geq 2$, $m=1$ and $\mathcal{Q}=0$, then $\mathcal{O}(\mathcal{V}^m_{f,\mathcal{Q}})$ coincides with the Cuntz algebra $\mathcal{O}_n$.

Exact sequences of C^*-algebras

Theorem

Let $\mathbf{B} = (B_1, \ldots, B_n)$ be the universal model of the noncommutative variety $\mathcal{V}_{f,\mathcal{Q}}^m$ and let $\mathcal{K}$ be the ideal of all compact operators in $B(\mathcal{N}_{f,\mathcal{Q}})$. Then the sequence of C^*-algebras

$$0 \to \mathcal{K} \to C^*(\mathbf{B}) \to \mathcal{O}(\mathcal{V}_{f,\mathcal{Q}}^m) \to 0$$

is exact.

Exact sequences of C^*-algebras

- If $n = 1$, $f = Z$, $m = 1$ and $\mathcal{Q} = 0$, we recover Coburn's theorem for $C^*(S)$:

$$0 \to \mathcal{K} \to C^*(S) \to C(\mathbb{T}) \to 0.$$

- If $f = Z_1 + \cdots + Z_n$, $n \geq 2$, $m = 1$ and $\mathcal{Q} = 0$, we obtain Cuntz exact sequence for the C^*-algebra generated by the left creation operators on the full Fock space :

$$0 \to \mathcal{K} \to C^*(S_1, \ldots, S_n) \to \mathcal{O}_n \to 0.$$

- If $f = Z_1 + \cdots + Z_n$, $m = 1$, and the ideal $\mathcal{Q} = \{Z_iZ_j - Z_jZ_i\}$, we recover Arveson's result for the C^*-algebra generated by the d-shift acting on the Drury-Arveson space :

$$0 \to \mathcal{K} \to C^*(M_{z_1}, \ldots, M_{z_n}) \to C(\partial\mathbb{B}_n) \to 0.$$

Exact sequences of C^*-algebras

One can obtain the following more general result

Theorem

If $\pi : C^*(\mathbf{B}) \to B(\mathcal{K})$ is an unital $*$-representation on a separable Hilbert space $\mathcal{K}$ such that π is not a Cuntz type representation, then the C^*-algebras $C^*(\mathbf{B})$ and $C^*(\pi(\mathbf{B}))$ are $*$-isomorphic and

$$0 \to \boldsymbol{\mathcal{K}} \to C^*(\pi(\mathbf{B})) \to \mathcal{O}(\mathcal{V}^m_{f,\mathcal{Q}}) \to 0$$

is a short exact sequence, where $\boldsymbol{\mathcal{K}}$ is the ideal of compact operators in $C^*(\pi(\mathbf{B}))$.

Commutative case

All our results apply in the particular case when f is a positive regular noncommutative polynomial and

$$\mathcal{Q} = \{Z_i Z_j - Z_j Z_i\}.$$

In this case, the corresponding C^*-algebra $C^*(\mathbf{B})$ coincides with the C^*-algebra generated by the multipliers $L_1, \ldots, L_n$ by the coordinate functions $\lambda_1, \ldots, \lambda_n$ and the identity, acting on the reproducing kernel Hilbert space $H^2(\mathcal{D}_f^\circ(\mathbb{C}))$ with kernel defined by

$$\kappa_f(\mu, \lambda) := \frac{1}{\left(1 - f(\mu_1 \bar{\lambda}_1, \ldots, \mu_n \bar{\lambda}_n)\right)^m}, \qquad \mu, \lambda \in \mathcal{D}_p^\circ(\mathbb{C}),$$

where

$$\mathcal{D}_f^\circ(\mathbb{C}) := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \ f(|\lambda_1|^2, \ldots, |\lambda_n|^2) < 1)\}.$$

Polydomains and varieties

- Let $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ be the set of all tuples $\mathbf{X} := (X_1, \ldots, X_k)$ in $B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ with the property that the entries of $X_s := (X_{s,1}, \ldots, X_{s,n_s})$ are commuting with the entries of $X_t := (X_{t,1}, \ldots, X_{t,n_t})$ for any $s, t \in \{1, \ldots, k\}$, $s \neq t$.
- Let $\mathbf{f} = (f_1, \ldots, f_k)$ be positive regular free holomorphic functions, $\mathbf{m} := (m_1, \ldots, m_k)$, $\mathbf{n} := (n_1, \ldots, n_k)$. The *regular polydomain* $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$ is the set of all k-tuples $\mathbf{X} = (X_1, \ldots, X_k) \in B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$ such that

$$\Delta_{\mathbf{f},\mathbf{X}}^{\mathbf{p}}(I) \geq 0 \quad \text{for} \quad \mathbf{0} \leq \mathbf{p} \leq \mathbf{m}, \quad \mathbf{p} := (p_1, \ldots, p_k) \in \mathbb{N}^k$$

where

$$\Delta_{\mathbf{f},\mathbf{X}}^{\mathbf{p}} := (id - \Phi_{f_1,X_1})^{m_1} \circ \cdots \circ (id - \Phi_{f_k,X_k})^{m_k}$$

Polydomains and varieties

- For each $i \in \{1, \ldots, k\}$, let $Z_i := (Z_{i,1}, \ldots, Z_{i,n_i})$ be an n_i-tuple of noncommutative indeterminates and assume that, for any $t, s \in \{1, \ldots, k\}$, $s \neq t$, the entries of Z_t are commuting with the entries if Z_s.
- We study noncommutative varieties in the polydomain $\mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H})$, given by

$$\mathcal{V}_{\mathbf{f},\mathcal{Q}}^{\mathbf{m}}(\mathcal{H}) := \{\mathbf{X} \in \mathbf{D}_{\mathbf{f}}^{\mathbf{m}}(\mathcal{H}) : \ g(\mathbf{X}) = 0 \text{ for all } g \in \mathcal{Q}\},$$

where $\mathcal{Q}$ is a set of polynomials in noncommutative indeterminates $Z_{i,j}$, which generates a nontrivial ideal in $\mathbb{C}[Z_{i,j}]$.

Polydomains and varieties

- Each variety $\mathcal{V}^{\mathbf{m}}_{\mathbf{f},\mathcal{Q}}$ admits a *universal model* $\mathbf{B} = \{\mathbf{B}_{i,j}\}$ such that $g(\mathbf{B}) = 0$, $g \in \mathcal{Q}$, acting on a subspace $\mathcal{N}_{\mathbf{f},\mathcal{Q}}$ of a tensor product of full Fock spaces, which is co-invariant under $\mathbf{B}_{i,j}$.

- The elements of the variety $\mathcal{V}^{\mathbf{m}}_{\mathbf{f},\mathcal{Q}}$ admit dilations which are $*$-representations of the C^*-algebra $C^*(\mathbf{B})$ generated by the universal model $\mathbf{B} = \{\mathbf{B}_{i,j}\}$ and the identity.

- Wold decompositions for the unital $*$-representations of the C^*-algebras $C^*(\mathbf{B})$ associated with the variety $\mathcal{V}^{\mathbf{m}}_{\mathbf{f},\mathcal{Q}}$ are obtained.

Polydomains and varieties

- The C^*-algebra $\mathcal{O}(\mathcal{V}_{\mathbf{f},\mathcal{Q}}^{\mathbf{m}})$ is the universal C^*-algebra generated by $V_{i,s} := \pi(\mathbf{B}_{i,s})$ and the identity, where π is a completely non-pure $*$-representation of the C^*-algebra $C^*(\mathbf{B})$, which is equivalent to

$$(id - \Phi_{f_k,V_k}) \circ \cdots \circ (id - \Phi_{f_1,V_1})(I_{\mathcal{K}}) = 0,$$

  where $V_i = (V_{i,1}, \ldots, V_{i,n_i})$.
- The sequence of C^*-algebras

$$0 \to \mathcal{K} \to C^*(\mathbf{B}) \to \mathcal{O}(\mathcal{V}_{\mathbf{f},\mathcal{Q}}^{\mathbf{m}}) \to 0$$

  is exact, where $\mathcal{K}$ denotes the ideal of compact operators in $B(\mathcal{N}_{\mathbf{f},\mathcal{Q}})$.

Polyvarieties

- Assume that $\mathcal{Q} = \cup_{i=1}^k \mathcal{Q}_i$ where $\mathcal{Q}_i$ is a set of polynomials in $\mathbb{C}\left\langle Z_{i,1}, \ldots Z_{i,n_i}\right\rangle$.The polyvariety associated is

$$\mathcal{V}_{f_1}^{m_1}(\mathcal{H}) \times \cdots \times \mathcal{V}_{f_k}^{m_k}(\mathcal{H}) := \mathcal{V}_{\mathbf{f},\mathcal{Q}}^{\mathbf{m}}(\mathcal{H}).$$

- The universal model satisfies both relations

$$\mathbf{B}_{i,s}\mathbf{B}_{j,t} = \mathbf{B}_{j,t}\mathbf{B}_{i,s} \quad \text{and} \quad \mathbf{B}_{i,s}\mathbf{B}_{j,t}^* = \mathbf{B}_{j,t}^*\mathbf{B}_{i,s}$$

for every $i, j \in \{1, \ldots, k\}$ with $i \neq j$, and every s, t.

- The C^*-algebra $C^*(\mathbf{B})$ generated by the operators $\mathbf{B}_{i,s}$ and the identity satisfies the relation

$$C^*(\mathbf{B}) = C^*(B_1) \otimes_{sp} \cdots \otimes_{sp} C^*(B_k),$$

where $C^*(B_i)$ is the C^*-algebra generated by $B_{i,1}, \ldots, B_{i,n_i}$ and the identity.

Refined Wold decomposition

Theorem

Any unital $*$-representation π of $C^*(\mathbf{B})$ on a separable Hilbert space $\mathcal{K}$ has a unique decomposition into a direct sum

$$\pi = \bigoplus_{A\subset\{1,\dots,k\}} \pi|_{\mathcal{K}_A} \quad \text{on} \quad \mathcal{K} = \bigoplus_{A\subset\{1,\dots,k\}} \mathcal{K}_A,$$

where $\{\pi|_{\mathcal{K}_A}\}_{A\subset\{1,\dots,k\}}$ are disjoint $*$-representations of $C^*(\mathbf{B})$, with the following properties :

(i) *for each $A \subset \{1,\dots,k\}$, the subspace $\mathcal{K}_A$ is reducing for every $\pi(\mathbf{B}_{i,j})$;*

(ii) *if $i \in A$, then $(\pi(\mathbf{B}_{i,1})|_{\mathcal{K}_A},\dots,\pi(\mathbf{B}_{i,n_i})|_{\mathcal{K}_A})$ is a pure tuple ;*

(iii) *if $i \in A^c$, then $(\pi(\mathbf{B}_{i,1})|_{\mathcal{K}_A},\dots,\pi(\mathbf{B}_{i,n_i})|_{\mathcal{K}_A})$ is a Cuntz tuple.*

Moreover, there is a unique wandering subspace $\mathcal{L}_A \subset \mathcal{K}_A$ that is reducing under the operators $\pi(\mathbf{B}_{j,t})$ for every $j \in A^c$ and $t \in \{1, \ldots, n_j\}$ such that $(\pi(\mathbf{B}_{j,1})|_{\mathcal{L}_A}, \ldots, \pi(\mathbf{B}_{j,n_j})|_{\mathcal{L}_A})$ is a *Cuntz* tuple and

$$\mathcal{K}_A = \mathcal{L}_A \bigoplus \overline{\text{span}} \left\{ \pi(\mathbf{B}_{i_1,\alpha_{i_1}}) \cdots \pi(\mathbf{B}_{i_p,\alpha_{i_p}})(\mathcal{L}_A) : \sum_{j=1}^{p} |\alpha_{i_j}| \geq 1 \right\},$$

where the wandering subspace is precisely described.

Classification

- For each subset $A \subset \{1, \ldots, k\}$, we introduce the *universal C^*-algebra $\mathcal{O}_A$* generated by a $*$-representation $\gamma : \otimes_{i \in A} C^*(B_i) \to B(\mathcal{K})$ with the the property that

$$\Phi_{f_i, \gamma_i(B_i)}(I_{\mathcal{N}_{J_i}}) = I_{\mathcal{N}_{J_i}} \quad \text{for every} \quad i \in A,$$

where $\gamma_i(B_i) := (\gamma_i(B_{i,1}), \ldots, \gamma_i(B_{i,n_i}))$.

Remark

In the particular case when $f_i = Z_{i,1} + \cdots + Z_{i,n_i}$ and $\mathcal{Q}_i = 0$, the algebra $\mathcal{O}_A$ coincides with the tensor algebra $\otimes_{i \in A} \mathcal{O}_{n_i}$, where $\mathcal{O}_{n_i}$ is the Cuntz algebra with n_i generators.

Classification

Theorem

There is a one-to-one correspondence between the unitary equivalence classes of the $*$-representations of the C^*-algebra $C^*(\mathbf{B})$ and the enumerations of 2^k unitary equivalence classes of $*$-representations of the universal algebras $\mathcal{O}_A$, as A is any subset of $\{1, \dots, k\}$.

Theorem

If $\mathcal{J}$ is the closed two-sided ideal of $C^*(\mathbf{B})$ generated by the orthogonal projections $I_{\mathcal{K}} - \Phi_{f_1,\mathbf{B}_1}(I)$, ..., $I_{\mathcal{K}} - \Phi_{f_k,\mathbf{B}_k}(I)$, then the following short sequence of C^*-algebras is exact

$$0 \to \mathcal{J} \to C^*(\mathbf{B}) \to \mathcal{O}_{\{1,\dots,k\}} \to 0.$$

Remark

If $\mathcal{Q}_i := \{Z_{i,s}Z_{i,t} - Z_{i,t}Z_{i,s}\}$, the universal model $\mathbf{L} = \{\mathbf{L}_{i,j}\}$ associated with $\mathcal{V}^{m_1}_{f_1,\mathcal{Q}_1} \times \cdots \times \mathcal{V}^{m_k}_{f_k,\mathcal{Q}_k}$, consists of the multipliers by the coordinate functions $\lambda_{i,j}$ acting on a Hilbert space $H^2(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathbb{C})^\circ)$ of holomorphic functions on the Reinhardt domain

$$\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathbb{C})^\circ = \mathcal{D}_{f_1}(\mathbb{C}^{n_1})^\circ \times \cdots \times \mathcal{D}_{f_k}(\mathbb{C}^{n_k})^\circ, \quad \text{where}$$

$$\mathcal{D}_{f_i}(\mathbb{C})^\circ = \{\lambda_i = (\lambda_{i,1}, \ldots, \lambda_{i,n_i}) \in \mathbb{C}^{n_i} : \ f_i(|\lambda_{i,1}|^2, \ldots, |\lambda_{i,n_i}|^2) < 1)\}.$$

More precisely, $H^2(\mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathbb{C})^\circ)$ is the reproducing kernel Hilbert space with kernel defined by

$$\kappa_{\mathbf{f}}(\mu, \lambda) := \frac{1}{\prod_{i=1}^k \left(1 - f_i(\mu_{i,1}\overline{\lambda}_{i,1}, \ldots, \mu_{i,n_i}\overline{\lambda}_{i,n_i})\right)^{m_i}}, \qquad \mu, \lambda \in \mathbf{D}^{\mathbf{m}}_{\mathbf{f}}(\mathbb{C})^\circ.$$

- All the results apply to the C^*-algebra $C^*(\mathbf{L}_{i,j})$ generated by the multipliers $\mathbf{L}_{i,j}$ and the identity.

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THANK YOU