Non-commutative rational multipliers of the free Hardy space

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NC rational functions

NC rational functions

Any valid combination, $\mathfrak{r}(\mathfrak{z})$ of NC variables $\mathfrak{z}_1, \cdots \mathfrak{z}_d$, $\mathbb{C}$, operations $+, \cdot, ^{-1}$ and $()$.

$$e.g. \quad \left(\mathfrak{z}_1\mathfrak{z}_2^{-1}\mathfrak{z}_1^2\right)^{-1} + 1 + \mathfrak{z}_1\mathfrak{z}_2.$$

$$\mathrm{Dom}\,\mathfrak{r} := \bigsqcup_{n=1}^{\infty} \left\{ Z \in \mathbb{C}_n^d := \mathbb{C}^{n\times n} \otimes \mathbb{C}^{1\times d} \middle| \ \mathfrak{r}(Z) \text{ is defined} \right\}$$

$$Z = (Z_1, \cdots, Z_d), \quad Z_k \in \mathbb{C}^{n\times n}$$

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Deep connections to: NC algebra, free real algebraic geometry and free probability.

[Helton, Kaliuzhnyĭ-Verbovetskyiĭ, Klep, McCullough, Pascoe, Vinnikov, Volčič,...]

NC rational functions

Realizations

Any NC rational $\mathfrak{r}$, $0 = (0, \cdots, 0) \in \mathrm{Dom}\, \mathfrak{r}$, has a *realization* (A, b, c), $A \in \mathbb{C}_N^d$, $b, c \in \mathbb{C}^N$: Given $Z \in \mathbb{C}_n^d$, $\mathfrak{r}(Z) = b^* L_A(Z)^{-1} c$:

$$L_A(Z) := I_n \otimes I_N - \sum_{j=1}^{d} Z_j \otimes A_j. \qquad \text{(Linear pencil)}$$

Minimal realizations

A realization (A, b, c) of size N is *minimal* if N is as small as possible.

$$(A, b, c) \text{ is minimal} \quad \Leftrightarrow \quad \bigvee A^{\omega} c = \mathbb{C}^N = \bigvee A^{*\omega} b$$

$$\omega = i_1 \cdots i_n, \quad i_k \in \{1, \cdots, d\}, \quad A^{\omega} := A_{i_1} \cdots A_{i_d}.$$

Any NC rational $\mathfrak{r}$ with $0 \in \mathrm{Dom}\, \mathfrak{r}$ has a (unique) minimal realization.

NC unit row–ball

Define $\mathbb{C}^d_{\mathbb{N}} := \bigsqcup_{n=1}^{\infty} \mathbb{C}^d_n$, $\mathbb{C}^d_n = \mathbb{C}^{n\times n} \otimes \mathbb{C}^{1\times d}$, *NC universe.*

$$\mathbb{B}^d_{\mathbb{N}} := \bigsqcup_{n=1}^{\infty} \mathbb{B}^d_n, \qquad \mathbb{B}^d_n := \left(\mathbb{C}^{n\times n} \otimes \mathbb{C}^{1\times d}\right)_1.$$

$Z = (Z_1, \cdots, Z_d) \in \mathbb{B}^d_n$ if $Z : \mathbb{C}^n \otimes \mathbb{C}^d \to \mathbb{C}^n$ has $\|Z\| < 1$,

$$ZZ^* = Z_1 Z_1^* + \cdots + Z_d Z_d^* < I_n.$$

finite–dimensional strict row contractions

Square-summable power series

The free Hardy space

$$\mathbb{H}^2_d := \left\{ f(\mathfrak{z}) = \sum_{\omega \in \mathbb{F}^d} \hat{f}_\omega \mathfrak{z}^\omega \,\middle|\, \sum |\hat{f}_\omega|^2 < \infty \right\},$$

$\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_d$; d NC variables.

$\mathbb{F}^d =$ all words in d letters, $\{1, 2, \cdots, d\}$. Empty word $= \emptyset$ (unit).

$$\textit{e.g.}\ \text{if } \omega = 1221 \in \mathbb{F}^2, \quad \mathfrak{z}^\omega := \mathfrak{z}_1\mathfrak{z}_2\mathfrak{z}_2\mathfrak{z}_1,\ \mathfrak{z}^\emptyset =: 1.$$

Hardy space

$$H^2 = \left\{ f(z) = \sum_{n \in \mathbb{N}_0} \hat{f}_n z^n \,\middle|\, \sum |\hat{f}_n|^2 < \infty \right\}.$$

Elements of $\mathbb{H}^2_d$ are NC functions on $\mathbb{B}^d_{\mathbb{N}}$:

(Popescu)

If $Z = (Z_1, \cdots, Z_d) \in \mathbb{B}^d_{\mathbb{N}}$ and

$$f(\mathfrak{z}) = \sum \hat{f}_\omega \mathfrak{z}^\omega \in \mathbb{H}^2_d$$

then $f(Z)$ converges absolutely (and locally uniformly) in operator-norm.

$\Rightarrow$ Any $f \in \mathbb{H}^2_d$ is a non-commutative function on strict row contractions.

Non-commutative functions

NC functions on the row-ball

Any $f \in \mathbb{H}^2_d$ is a *non-commutative function* on $\mathbb{B}^d_{\mathbb{N}}$, *i.e.* f

- is graded,
- preserves direct sums,
- respects joint similarities which preserve $\mathbb{B}^d_{\mathbb{N}}$,

locally bounded hence holomorphic.

Any free polynomial, $p \in \mathbb{C}\{\mathfrak{z}_1, \cdots, \mathfrak{z}_d\}$ belongs to $\mathbb{H}^2_d$, *e.g.*
$p(\mathfrak{z}_1, \mathfrak{z}_2) = \mathfrak{z}_1^2 \mathfrak{z}_1 \mathfrak{z}_2 - 5\mathfrak{z}_1^3 \mathfrak{z}_2^2$.

NC Hardy Space

H^∞ = bounded holomorphic functions:

$$H^\infty = \left\{ f \in \mathscr{O}(\mathbb{D}) \;\middle|\; \|f\|_\infty = \sup_{z \in \mathbb{D}} \|f(z)\| < \infty \right\}.$$

NC Hardy Space

NC H^∞ = bounded NC holomorphic functions:

$$\mathbb{H}^\infty_d = \left\{ F \in \mathscr{O}(\mathbb{B}^d_{\mathbb{N}}) \;\middle|\; \|F\|_\infty = \sup_{Z \in \mathbb{B}^d_{\mathbb{N}}} \|F(Z)\| < \infty \right\}.$$

$$F \in \mathbb{H}^\infty_d \;\Leftrightarrow F(L) := M^L_F : \mathbb{H}^2_d \to \mathbb{H}^2_d, \quad \text{bounded left multiplier,}$$

$$(M^L_F f)(Z) = F(Z) f(Z); \qquad Z \in \mathbb{B}^d_{\mathbb{N}}.$$

The left free shift

$L_k := M^L_{Z_k} : \mathbb{H}^2_d \to \mathbb{H}^2_d$, isometries, pairwise orthogonal ranges.

$$\Rightarrow \; L := (L_1, \cdots, L_d) : \mathbb{H}^2_d \otimes \mathbb{C}^d \to \mathbb{H}^2_d, \quad \text{is a } \textit{row isometry}.$$

Left vs. right

letter reversal $\mathrm{t} : \mathbb{F}^d \to \mathbb{F}^d$

$$\begin{aligned} \omega &= i_1 \cdots i_n \mapsto \omega^{\mathrm{t}} := i_n \cdots i_1, \qquad \textit{involution.} \\ U_{\mathrm{t}} &: \mathbb{H}^2_d \to \mathbb{H}^2_d, \qquad \mathfrak{z}^{\omega} \overset{U_{\mathrm{t}}}{\mapsto} \mathfrak{z}^{\omega^{\mathrm{t}}} \quad \textit{unitary involution.} \end{aligned}$$

Right multipliers

$R_k := M^R_{Z_k} = U_{\mathrm{t}} M^L_{Z_k} U_{\mathrm{t}} = U_{\mathrm{t}} L_k U_{\mathrm{t}}$, *right free shifts.*

Given $p \in \mathbb{C}\{\mathfrak{z}_1, \cdots, \mathfrak{z}_d\}$, $p(L) = M^L_p$, $p(R) = M^R_{p^{\mathrm{t}}}$.

$$\textit{e.g.} \quad p(\mathfrak{z}) = 1 - \mathfrak{z}_1\mathfrak{z}_2 \mapsto p^{\mathrm{t}}(\mathfrak{z}) = 1 - \mathfrak{z}_2\mathfrak{z}_1.$$

If $b \in \mathbb{H}^\infty_d$, $b(R) := U_{\mathrm{t}} b(L) U_{\mathrm{t}} = M^R_{b^{\mathrm{t}}}$.

NC rational functions in Fock space

Fock space as an NC-RKHS

$\mathbb{H}^2_d$ is an NC reproducing kernel Hilbert space on $\mathbb{B}^d_{\mathbb{N}}$ [Ball–Marx–Vinnikov].

$$f \stackrel{\ell_{Z,y,v}}{\mapsto} y^* f(Z) v \,; \qquad Z, y, v \in \mathbb{B}^d_n \times \mathbb{C}^n \times \mathbb{C}^n,$$

is bounded on $\mathbb{H}^2_d$.

$$\ell_{Z,y,v}(f) = \langle K\{Z,y,v\}, f \rangle_{\mathbb{H}^2}, \quad K\{Z,y,v\} \in \mathbb{H}^2_d.$$

NC Szegö kernel vectors

$$\begin{aligned} K\{Z,y,v\} &= \sum \overline{y^* Z^\alpha v}\, \mathfrak{z}^\alpha, \quad Z \in \mathbb{B}^d_n, \\ \langle K\{Z,y,v\}, f \rangle_{\mathbb{H}^2} &= \sum_{\alpha,\omega} y^* Z^\alpha v\, \hat{f}_\omega \underbrace{\langle \mathfrak{z}^\alpha, \mathfrak{z}^\omega \rangle}_{=\delta_{\alpha,\omega}}; \qquad f = \sum \hat{f}_\omega \mathfrak{z}^\omega \in \mathbb{H}^2_d \\ &= y^* \sum \hat{f}_\omega\, Z^\omega v = y^* f(Z) v. \end{aligned}$$

NC kernels are rational

Any NC Szegö kernel, $K\{W,x,u\} \in \mathbb{H}^2_d$, $W \in \mathbb{B}^d_m$ is given by:

$$K\{W,x,u\}(Z) = \sum_{\omega} \overline{x^* W^{\omega} u}\, Z^{\omega}; \qquad Z \in \mathbb{B}^d_n$$

$$= \sum \overline{x}^* \overline{W}^{\omega} \overline{u}\, Z^{\omega}; \qquad \overline{x}, \overline{W}, \overline{u} := \text{entry–wise conjugation}$$

$$= \overline{x}^* \otimes I_n \sum_{k=0}^{\infty} \left(\sum_{j=1}^{d} \overline{W}_j \otimes Z_j \right)^k \overline{u} \otimes I_n; \qquad \mathrm{spr}\,(\cdot) < 1$$

$$= \overline{x}^* \otimes I_n \left(I_m \otimes I_n - \sum \overline{W}_j \otimes Z_j\right)^{-1} \overline{u} \otimes I_n$$

$$= \boxed{\overline{x}^* L_{\overline{W}}(Z)^{-1} \overline{u}}, \qquad \text{NC rational!}$$

NC rational functions in $\mathbb{H}^2_d$ are NC kernels

If $\mathfrak{r} \in \mathbb{H}^2_d$ NC rational with minimal realization (A, b, c) of size N,

$$\begin{aligned}
\mathfrak{r}(Z) &= \sum_{\omega} \hat{\mathfrak{r}}_\omega Z^\omega \qquad Z \in \mathbb{B}^d_n \\
&= I_n \otimes b^* \left(I_n \otimes I_N - \sum Z_j \otimes A_j \right)^{-1} I_n \otimes c \\
\text{`}&= \sum \underbrace{b^* A^\omega c}_{=\hat{\mathfrak{r}}_\omega} Z^\omega = K\{\overline{A}, \overline{b}, \overline{c}\}(Z)\text{'},
\end{aligned}$$

i.e. $\mathfrak{r}$ formally resembles an NC Szegö kernel.

NC rational functions in $\mathbb{H}^2_d$ are NC kernels

Definition (Joint spectral radius – Popescu)

Given $A = (A_1, \cdots, A_d) \in \mathbb{C}^d_N$,

$$\mathrm{spr}(A) := \lim \sqrt[2k]{\left\| \mathrm{Ad}^{(k)}_{A,A^*}(I_N) \right\|}; \quad \mathrm{Ad}_{A,A^*}(P) = A(P \otimes I_d)A^*.$$

Multi-variable Rota–Strang Theorem (Popescu)

$A = (A_1, \cdots, A_d) \in \mathbb{C}^d_N$ has $\mathrm{spr}(A) < 1$ if and only if A is jointly similar to a strict row contraction $W \in \mathbb{B}^d_N$.

If NC rational $\mathfrak{r} \in \mathbb{H}^2_d$ with minimal (A, b, c) can show $\mathrm{spr}(A) < 1$.
Multi-variable Rota Theorem $\Rightarrow$ $\mathfrak{r} = K\{Z, \widetilde{b}, \widetilde{c}\}$ for $Z \in \mathbb{B}^d_{\mathbb{N}}$ similar to $\overline{A}$.

Theorem (Jury–M.–Shamovich)

Let $\mathfrak{r}$ be an NC rational function with $0 \in \mathrm{Dom}\,\mathfrak{r}$ and minimal realization (A, b, c) of size N. The following are equivalent:

(i) $\mathfrak{r} \in \mathbb{H}^2_d$.

(ii) $\mathfrak{r} \in \mathbb{H}^\infty_d$.

(iii) $\mathfrak{r} \in \mathbb{A}_d := \mathrm{Alg}\{I, L_1, \cdots, L_d\}^{-\|\cdot\|}$ *(NC disk algebra).*

(iv) $\mathrm{spr}(A) < 1$.

(v) $\mathfrak{r} = K\{Z, y, v\}$ *for some* $Z \in \mathbb{B}^d_N$ *similar to* $\overline{A}$.

(vi) $\overline{\mathbb{B}^d_{\mathbb{N}}} \subset \mathrm{Dom}\,\mathfrak{r}$.

(vii) *There exists an* $r > 1$ *so that* $r\mathbb{B}^d_{\mathbb{N}} \subseteq \mathrm{Dom}\,\mathfrak{r}$.

$d = 1$

This recovers (obvious) facts for rational functions in one variable.

Observe:

If $0 \in \mathrm{Dom}\,\mathfrak{r}$, NC rational with minimal (A, b, c), $A \in \mathbb{C}_n^d$ we can re-scale, $\mathfrak{r}_r(Z) := \mathfrak{r}(rZ)$, $r > 0$, to obtain $\mathbb{B}_{\mathbb{N}}^d \subseteq \mathrm{Dom}\,\mathfrak{r}_r$,

$$r^{-1} > \|A\|_{row} \geq \mathrm{spr}(A) \quad \Rightarrow \mathfrak{r}_r \in \mathbb{H}_d^\infty.$$

$\Rightarrow$ We can assume any $\mathfrak{r}$ which is regular at 0 belongs to $[\mathbb{H}_d^\infty]_1$.

Hardy space techniques can be applied to NC rational function theory.

Finite NC Blaschke products

Any $h \in H^\infty$ has a unique inner–outer factorization,

$$h = \theta \cdot f, \qquad \theta, f \in H^\infty,$$

where

- θ is *inner*, *i.e.* $M_\theta : H^2 \to H^2$ is isometric.
- f is *outer*, *i.e.* M_f has dense range.

Rational $\mathfrak{r} \in H^\infty$ is *inner* if and only if $\mathfrak{r}$ is a finite *Blaschke product*:

$$\mathfrak{r}(z) = \zeta \prod_{k=1}^{N} \frac{z - z_k}{1 - \overline{z_k} z}, \qquad z_k \in \mathbb{D},\ \zeta \in \partial\mathbb{D}.$$

Blaschke products are completely determined by their zeroes in the disk.

Inner and Outer

Definition

$h \in \mathbb{H}^\infty_d$ is:

1. *inner* if $h(L) = M^L_h : \mathbb{H}^2_d \to \mathbb{H}^2_d$ is an isometry.
   inner $=$ isometric
2. *outer* if $h(L) = M^L_h : \mathbb{H}^2_d \to \mathbb{H}^2_d$ has dense range.
   outer $=$ dense range

Theorem (Popescu, Davidson–Pitts)

Any $h \in \mathbb{H}^\infty_d$ (or $\mathbb{H}^2_d$) has a unique inner–outer factorization, $h = \Theta \cdot F$.

Inner–Outer factorization

Theorem (Jury–M.–Shamovich)

If $\mathfrak{r} \in \mathbb{H}^\infty_d$ is NC rational with inner–outer factorization $\mathfrak{r} = \Theta \cdot F$, then both $\Theta, F \in \mathbb{H}^\infty_d$ are NC rational.

Proof.

Recall: If h is a multiplier of an RKHS $\mathcal{H}(k)$, $M_h^* k_z = k_z \overline{h(z)}$.

$$\text{Similarly, if } h \in \mathbb{H}^\infty_d, \quad h(L)^* K\{Z, y, v\} = K\{Z, h(Z)^* y, v\}.$$

If $\mathfrak{r} = \Theta \cdot F \in \mathbb{H}^\infty_d$ then $\mathfrak{r} = K\{Z, y, v\}$,

$$K\{Z, \Theta(Z)^* y, v\} = \Theta(L)^* \mathfrak{r} = \underbrace{\Theta(L)^* \Theta(L)}_{=I} F.$$

$\Rightarrow$ F is an NC kernel (NC rational) and so is $\Theta = \mathfrak{r} F^{-1}$. $\square$

Finite NC Blaschke products

Definition (Jury–M.–Shamovich)

Given $h \in \mathbb{H}^2_d$, the *NC variety* of *h* is

$$\mathscr{Z}(h) := \bigsqcup_{\mathbb{N}\cup\{\infty\}} \left\{ (Z, y) \in \mathbb{B}^d_n \times \mathbb{C}^n \setminus \{0\} \middle| \; y^* h(Z) = 0 \right\}.$$

An inner $b \in \mathbb{H}^\infty_d$ is *NC Blaschke* if $\mathrm{Ran}\, \mathfrak{b}(L) = \mathscr{D}(\mathfrak{b})$,

$$\mathscr{D}(\mathfrak{b}) := \left\{ h \in \mathbb{H}^2_d \,|\, (Z, y) \in \mathscr{Z}(b) \Rightarrow (Z, y) \in \mathscr{Z}(h) \right\}.$$

i.e. NC Blaschke inners are completely determined by their 'NC zeroes'.

Theorem (Jury–M.–Shamovich)

Any NC rational inner $\mathfrak{b} \in \mathbb{H}^\infty_d$ is NC Blaschke.

Definition

An NC rational inner, $\mathfrak{b}$, is an *NC Blaschke factor*, if it is *irreducible*: If $\mathfrak{b} = \mathfrak{b}_1 \cdot \mathfrak{b}_2$, $\mathfrak{b}_k$ NC rational inner, then $\{\mathfrak{b}_1, \mathfrak{b}_2\} = \{\mathfrak{b}, 1\}$.

$d = 1$: $\mathfrak{b}$ is irreducible $\Leftrightarrow \mathfrak{b}(z) = \dfrac{z - w}{1 - \overline{w}z}$, a single *Blaschke factor*.

Theorem (M.)

Any NC rational inner $\mathfrak{b} \in \mathbb{H}_d^\infty$ is a finite NC Blaschke product, $\mathfrak{b} = \mathfrak{b}_1 \cdots \mathfrak{b}_n$, with each $\mathfrak{b}_k$ an NC rational Blaschke factor.

Proof idea:

If $\mathfrak{b} = \mathfrak{b}_1 \cdot \mathfrak{b}_2$ where $\mathfrak{b}_k$ are inner, then the minimal realization size of $\mathfrak{b}$ is greater than that of $\mathfrak{b}_1$.

Q: Can we characterize NC Blaschke factors in terms of their minimal realizations à la Helton–Klep–Volčič?

NC Clark Theory

Clark Theory

A bijection ($d = 1$)

μ	$\leftrightarrow$	H_μ	$\leftrightarrow$	b_μ
≥ 0		$\operatorname{Re} H_\mu(z) \geq 0$		$\lvert b_\mu(z)\rvert \leq 1$
Finite, positive, regular Borel measure		Herglotz function		contractive multiplier
		$H_\mu(z) := \int_{\partial\mathbb{D}} \frac{1+z\overline{\zeta}}{1-z\overline{\zeta}}\,\mu(d\zeta)$		$b_\mu(z) := \frac{H_\mu(z)-1}{H_\mu(z)+1}$

If $\mu \leftrightarrow b$ then $\mu = \mu_b$ is the *Aleksandrov–Clark measure of* b.

NC measures

Riesz–Markov: $\mathrm{Meas}(\partial\mathbb{D})_+ \ \leftrightarrow\ \mathscr{C}(\partial\mathbb{D})_+^\dagger = (\mathcal{A}(\mathbb{D}) + \mathcal{A}(\mathbb{D})^*)_+^\dagger$,

$\mu \ \leftrightarrow\ \hat{\mu}$, $\mathcal{A}(\mathbb{D}) = \mathrm{Alg}\{I, S\}^{-\|\cdot\|}$ Disk Algebra.

$S = M_z : H^2 \to H^2$, the *shift*.

$\mathbb{A}_d := \mathrm{Alg}\{I, L_1, \cdots, L_d\}^{-\|\cdot\|}$ *Free Disk Algebra*.

Definition

A (positive) NC measure, $\mu \in (\mathscr{A}_d^\dagger)_+$, is a positive linear functional on the *Free Disk System*, $\mathscr{A}_d := (\mathbb{A}_d + \mathbb{A}_d^*)^{-\|\cdot\|}$.

Given any positive NC measure, $\mu : \mathscr{A}_d \to \mathbb{C}$, consider:

NC Herglotz Function

$$\begin{aligned} H_\mu(Z) &= \mathrm{id}_n \otimes \mu\left((I + ZL^*)(I - ZL^*)^{-1}\right), \\ ZL^* &:= Z_1 \otimes L_1^* + \ldots + Z_d \otimes L_d^*, \quad Z \in \mathbb{B}^d_{\mathbb{N}} \\ & \quad\ \mathrm{Re}\, H_\mu(Z) \geq 0; \qquad Z \in \mathbb{B}^d_{\mathbb{N}}, \end{aligned}$$

$$b_\mu(Z) := (H_\mu(Z) + I_n)^{-1}(H_\mu(Z) - I_n) \in [\mathbb{H}^\infty_d]_1.$$

A bijection

$$\begin{array}{ccccc} \mu & \leftrightarrow & H_\mu & \leftrightarrow & b_\mu \\ \\ \geq 0 & & \mathrm{Re}\, H_\mu \geq 0 & & \|b_\mu(Z)\| \leq 1 \end{array}$$

If $\mu \leftrightarrow b$ then $\mu = \mu_b$ is the *NC Clark measure of* b.

NC measure spaces

$$\mathbb{H}^2_d(\mu) := \mathbb{C}\left\{\mathfrak{z}_1, \cdots, \mathfrak{z}_d\right\}^{-\|\cdot\|_\mu}; \quad \langle p, q\rangle_\mu := \mu(\underbrace{p(L)^* q(L)}_{\in \mathscr{A}_d}).$$

Gelfand-Naimark-Segal row isometry

$$M^L_{\mathfrak{z}_k}\, p(\mathfrak{z}) + N_\mu = \mathfrak{z}_k p(\mathfrak{z}) + N_\mu; \qquad \boxed{\Pi_{\mu;k} = M^L_{\mathfrak{z}_k}}$$

N_μ :=ideal of '$0-$length vectors'. $\Pi_\mu = (\Pi_{\mu;1}, \cdots, \Pi_{\mu;k})$ row isometry.

Extends to $*-$representation of $C^*\{I, L\} =: \mathcal{E}_d$, *Cuntz-Toeplitz algebra*.

If $d = 1$

$$\mathbb{H}^2_1(\mu) \simeq H^2(\mu) = \bigvee_{n\geq 0} \zeta^n \subseteq L^2(\mu), \qquad \Pi_\mu \simeq M_\zeta|_{H^2(\mu)} = \text{ mult. by } \zeta.$$

NC rational Clark measures

Finitely–correlated Cuntz-Toeplitz functionals

$\mu \in (\mathscr{A}_d^\dagger)_+$ is *finitely–correlated* if the subspace

$$\mathscr{H}_\mu := \bigvee \Pi_\mu^{*\omega} 1 + N_\mu \subseteq \mathbb{H}^2_d(\mu),$$

is finite–dimensional.

Dilation of finite row contractions

If μ is finitely–correlated then Π_μ is the minimal row isometric dilation of the finite–dimensional row contraction $T_\mu := (\Pi_\mu^*|_{\mathscr{H}_\mu})^*$.

– Finitely–correlated states on the Cuntz algebra were introduced and studied by Bratteli and Jørgensen.

– Row isometric dilations of finite row contractions were completely classified by Davidson–Kribs–Shpigel.

If $\mu \in (\mathscr{A}_d^\dagger)_+$ is finitely correlated, then,

$$\begin{aligned}\mathrm{id}_n \otimes \mu\left((I - ZL^*)^{-1}\right) &= (1 + N_\mu)^*(I - ZT_\mu^*)^{-1}(1 + N_\mu) \\ &= \frac{1}{2}H_\mu(Z) + \frac{1}{2}\mu(I)I_n =: G_\mu(Z).\end{aligned}$$

Recall: $ZL^* = Z_1 \otimes L_1^* + \cdots + Z_d \otimes L_d^*$.

Let $x := 1 + N_\mu$, and $T_\mu := \left(\Pi_\mu^*|_{\mathscr{H}_\mu}\right)^*$, $\dim \mathscr{H}_\mu < +\infty$.

$\left(T_\mu^*, x, x\right)$ is a finite–dimensional (and minimal) realization of G_μ.

$$\Rightarrow \quad H_\mu \quad \text{and} \quad \mathfrak{b}_\mu = (H_\mu - I)(H_\mu + I)^{-1} \qquad \text{are NC rational.}$$

Theorem (M.)

A contractive $\mathfrak{b} \in [\mathbb{H}^\infty_d]_1$ *is NC rational if and only if* $\mu_{\mathfrak{b}} \in (\mathscr{A}_d^\dagger)_+$ *is finitely–correlated.*

NC rational extreme points of $[\mathbb{H}_d^\infty]_1$

de Branges–Rovnyak realizations

Any $b \in [\mathbb{H}^\infty_d]_1$ has a *de Branges–Rovnyak realization*:

(right) Free de Branges–Rovnyak space

$\mathscr{H}^{\mathrm{t}}(b) := \mathrm{Ran}\,\sqrt{I - b(R)b(R)^*}$, an NC-RKHS contained (contractively) in $\mathbb{H}^2_d$ with CPNC kernel:

$$K^b(Z,W)[\cdot] = K(Z,W)[\cdot] - K(Z,W)[b^{\mathrm{t}}(Z)(\cdot)b^{\mathrm{t}}(W)^*], \quad K = \text{NC Szegö}.$$

de Branges–Rovnyak realization

$$b(Z) \;=\; D \cdot I_n + C(I_n \otimes I - \sum Z_j \otimes A_j)^{-1} \sum_k Z_k \otimes B_k,$$

$$A := L^*|_{\mathscr{H}^{\mathrm{t}}(b)}, \quad B := L^* b^{\mathrm{t}}, \quad C := (K_0^b)^* \quad \text{and} \quad D := b(0).$$

Column-extreme multipliers of Fock space

$b \in [\mathbb{H}^\infty_d]_1$ is *column-extreme (CE)*, if $a \in \mathbb{H}^\infty_d$ and $c := \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right) \in [\mathbb{H}^\infty_d \otimes \mathbb{C}^2]_1$ implies $a \equiv 0$.

Note: $c := \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)$ is contractive if and only if:

$$I - b(R)^* b(R) \geq a(R)^* a(R) > 0.$$

c is inner if and only if

$$I - b(R)^* b(R) = a(R)^* a(R),$$

$T = I - b(R)^* b(R)$ is *factorizable*.

Sarason's outer function

– In one variable, $b \in [H^\infty]_1$ is column-extreme $\Leftrightarrow$ it is an extreme point.

– If b is NOT extreme, there is a unique outer, $a \in [H^\infty]_1$ so that $c = \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)$ is inner (isometric).

Sarason function

$c := \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)$ is defined by the colligation, $U_c : \mathscr{H}(b) \oplus \mathbb{C}^2 \to \mathscr{H}(b) \oplus \mathbb{C}^2$:

$$U_c := \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}; \qquad \begin{pmatrix} b(z) \\ a(z) \end{pmatrix} := \mathbf{D} + \mathbf{C}(I - z\mathbf{A})^{-1} z\mathbf{B},$$

$$\mathscr{H}(b) = \mathrm{Ran}\,\sqrt{I - b(S)b(S)^*} \quad \text{de Branges-Rovnyak space}$$

$$\boxed{S := M_z : H^2 \to H^2} \ \textit{the shift}, \quad a(0)^2 := 1 - |b(0)|^2 - \|S^* b\|^2_{\mathscr{H}(b)} > 0.$$

$$\mathbf{A} := S^*|_{\mathscr{H}(b)}, \quad \mathbf{B} := S^* b, \quad \mathbf{C} := \begin{pmatrix} (K_0^b)^* \\ -a(0)\langle b, \cdot \rangle_{\mathscr{H}(b)} \end{pmatrix} \quad \mathbf{D} := \begin{pmatrix} b(0) \\ a(0) \end{pmatrix}.$$

Theorem (Jury–M.)

Given any $b \in [\mathbb{H}_d^\infty]_1$, let $a \in [\mathbb{H}_d^\infty]_1$ be its NC Sarason function.

(i) *If b is non-CE then a is outer, $a(R)^* a(R)$ is the maximal factorizable minorant of $I - b(R)^* b(R)$.*

(ii) *$c := \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)$ is column-extreme.*

(iii) *c is inner if and only if b is non-CE and*

$$\sum_{|\omega|=n} \| L^{*\omega} b^{\mathrm{t}} \|^2_{\mathscr{H}^{\mathrm{t}}(b)} \to 0 \qquad \textit{weak purity}\ ^{(*)}. \tag{1}$$

(iv) *b is inner if and only if it is CE and (1) holds.*

$^{(*)}$ $T : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ is *pure* if $\sum_{|\omega|=n} \| T^{*\omega} h \|^2_{\mathcal{H}} \to 0$ for any $h \in \mathcal{H}$.

$X^* = (L^*|_{\mathscr{H}^{\mathrm{t}}(b)})^*$ pure $\Leftrightarrow$ b inner (Ball–Bolotnikov–Fang).

Corollary (Jury–M.)

If $\mathfrak{b} \in [\mathbb{H}^\infty_d]_1$ *is NC rational, then* $\mathfrak{b}$ *is either non-CE or inner.*

If $\mathfrak{b}$ *is non-CE, then the Sarason* $\mathfrak{a}$ *is NC rational and* $\mathfrak{c} := \begin{pmatrix} \mathfrak{b} \\ \mathfrak{a} \end{pmatrix}$ *is inner.*

▸ skip Proof

Corollary (Jury–M.)

If $\mathfrak{b} \in [\mathbb{H}^\infty_d]_1$ is NC rational, then $\mathfrak{b}$ is either non-CE or inner.

If $\mathfrak{b}$ is non-CE, then the Sarason $\mathfrak{a}$ is NC rational and $\mathfrak{c} := \left(\begin{smallmatrix} \mathfrak{b} \\ \mathfrak{a} \end{smallmatrix}\right)$ is inner.

Proof idea:

If $\mathfrak{b} \in \mathbb{H}^\infty_d$ is NC rational, so is $\mathfrak{b}^{\mathrm{t}} = K\{Z, y, v\}$.

$$\mathscr{M} := \bigvee_{\omega \neq \emptyset} L^{*\omega} \mathfrak{b}^{\mathrm{t}} = \bigvee_{\omega \neq \emptyset} K\{Z, Z^{*\omega} y, v\} \subseteq \mathscr{H}^{\mathrm{t}}(\mathfrak{b}).$$

'Restrict' the de Branges–Rovnyak realization of $\mathfrak{b}$ to $\mathscr{M}$:

$$\mathfrak{A} := L^*|_{\mathscr{M}}, \quad \mathfrak{B} = B = L^* \mathfrak{b}^{\mathrm{t}}, \quad \mathfrak{C} = (P_{\mathscr{M}} K_0^{\mathfrak{b}})^* \quad \text{and} \quad \mathfrak{D} = D = \mathfrak{b}(0),$$

$$\mathfrak{b}(Z) = \mathfrak{D} \cdot I_n + I_n \otimes \mathfrak{C} \left(I_n \otimes I - \sum Z_j \otimes \mathfrak{A}_j\right)^{-1} \sum Z_j \otimes \mathfrak{B}_j.$$

Restricting the realization for Sarason function $\mathfrak{a}$ gives a finite–dimensional Fornasini–Marchesini (FM) realization for $\mathfrak{a}$. $\Rightarrow$ $\mathfrak{a}$ is also NC rational.

Corollary (Jury–M.)

If $\mathfrak{b} \in [\mathbb{H}^\infty_d]_1$ is NC rational, then $\mathfrak{b}$ is either non-CE or inner.
If $\mathfrak{b}$ is non-CE, then the Sarason $\mathfrak{a}$ is NC rational and $\mathfrak{c} := \left(\begin{smallmatrix} \mathfrak{b} \\ \mathfrak{a} \end{smallmatrix}\right)$ is inner.

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If (A, b, c) is a minimal descriptor realization $\mathrm{spr}(A) < 1 \Rightarrow \mathrm{spr}(\mathfrak{A}) < 1$.

$\Rightarrow$ the weak purity condition (1) of the previous theorem holds.

Theorem (Fejér–Riesz)

Let $\operatorname{Re} p(S) \geq 0$, $p \in \mathbb{C}[z]$, *'positive trigonometric polynomial'. Then* $\operatorname{Re} p(S) = q(S)^* q(S)$ *for some* $q \in \mathbb{C}[z]$.

Here, $S := M_z : H^2 \to H^2$, the *shift*.

Theorem (NC Fejér–Riesz. Popescu, McCullough)

Suppose $p \in \mathbb{C}\{\mathfrak{z}_1, \cdots, \mathfrak{z}_d\}$ *and* $\operatorname{Re} p(R) \geq 0$. *Then there is a free polynomial* q *so that* $\operatorname{Re} p(R) = q(R)^* q(R)$.

Theorem (NC rational Fejér–Riesz)

Let $T := \operatorname{Re} \widetilde{\mathfrak{r}}(R) \geq 0$, *where* $\widetilde{\mathfrak{r}} \in \mathbb{H}^\infty_d$ *is NC rational. Then* T *factors as* $T = \mathfrak{r}(R)^* \mathfrak{r}(R)$ *for some NC rational outer* $\mathfrak{r} \in \mathbb{H}^\infty_d$.

skip Proof

Proof idea:

Classically, given $|f| > 0$, $|f| \in L^\infty$, consider $\mu := |f| \cdot m$, $m =$Lebesgue.

$$\mu = \mu_b, \;\; b \in [H^\infty]_1 \;\; \Rightarrow \;\; |f(\zeta)| = \frac{\mu_b(d\zeta)}{m(d\zeta)} = \frac{1 - |b(\zeta)|^2}{|1 - b(\zeta)|^2} \quad a.e. \quad \text{(Fatou)}$$

Let $\mu_T(L^\omega) := \langle 1, TL^\omega 1\rangle_{\mathbb{H}^2}$, $\mu_T \in (\mathscr{A}_d^\dagger)_+$ with 'NC Radon–Nikodym derivative' $T = \mathrm{Re}\,\widetilde{\mathfrak{r}}(R) \geq 0$.

By the NC Fatou Theorem (Jury-M.), if $\mu_T = \mu_{\mathfrak{b}}$,

$$\mathrm{Re}\,\widetilde{\mathfrak{r}}(R) = (I - \mathfrak{b}(R)^*)^{-1}(I - \mathfrak{b}(R)^*\mathfrak{b}(R))(I - \mathfrak{b}(R))^{-1},$$

for some NC rational $\mathfrak{b} \in [\mathbb{H}_d^\infty]_1 \Rightarrow I - \mathfrak{b}(R)^*\mathfrak{b}(R) = \mathfrak{a}(R)^*\mathfrak{a}(R)$.

d=1

If $\mathfrak{r}(z) = \dfrac{p(z)}{q(z)} \in H^\infty$, then

$$q(S)^* \mathrm{Re}\, \mathfrak{r}(S) q(S) = p(S)^* q(S) + q(S)^* p(S) = \mathrm{Re}\, \widetilde{q}(S) \geq 0, \qquad \widetilde{q} \in \mathbb{C}[z].$$

In one variable the rational Fejér–Riesz theorem is trivial.

$\Rightarrow$ any rational $\mathfrak{b} \in [H^\infty]_1$ is either inner or not an extreme point.

Theorem (M.)

Given NC rational $\mathfrak{b} \in [\mathbb{H}_d^\infty]_1$, *the following are equivalent:*

1. $\mathfrak{b}$ *is inner.*
2. $\Pi_{\mu_\mathfrak{b}}$ *is a Cuntz (onto) row isometry (of dilation–type, Kennedy).*
3. $T_\mu = \left(\Pi_\mu^*|_{\mathscr{H}_\mu}\right)^*$, $\mu = \mu_\mathfrak{b}$, *is a finite row co-isometry.*
4. $\mu_\mathfrak{b}$ *is a singular NC measure (Jury–M.)*

$d = 1$

$b \in H^\infty$ is rational inner $\Rightarrow$ b is a finite Blaschke product:

$$b(z) = z^n \prod_{k=1}^{N} \frac{z - z_k}{1 - \overline{z_k} z}; \qquad z_k \in \mathbb{D}.$$

μ_b is a finite, positive sum of point masses on $\partial\mathbb{D}$.
$\Rightarrow$ μ_b is singular and $M_\zeta|_{H^2(\mu_b)}$ is unitary.

Theorem (M.–Shamovich)

An NC rational function $\mathfrak{b}$ with $0 \in \operatorname{Dom} \mathfrak{b}$ belongs to $[\mathbb{H}_d^\infty]_1$ if and only it has a minimal Fornasini–Marchesini realization $U = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$:

$$\mathfrak{b}(Z) = D \cdot I_n + I_n \otimes C(I_n \otimes I - \sum Z_j \otimes A_j)^{-1} \sum Z_j \otimes B_j.$$

Every NC rational $\mathfrak{b}$ with $0 \in \operatorname{Dom} \mathfrak{b}$ has a minimal FM realization.

$$\textit{minimal FM realizations } \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \leftrightarrow \textit{ minimal descriptor } (A', b, c).$$

Theorem (M.–Shamovich)

An NC rational function $\mathfrak{b}$ *with* $0 \in \mathrm{Dom}\,\mathfrak{b}$ *belongs to* $[\mathbb{H}^\infty_d]_1$ *if and only it has a minimal Fornasini–Marchesini realization* $U = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$*:*

$$U := \begin{pmatrix} T_0^* & (1-\overline{\mathfrak{b}(0)})\,T^*x \\ (1-\mathfrak{b}(0))\langle P_{\mathcal{H}_0}x, \cdot\rangle_{\mathcal{H}_0} & \mathfrak{b}(0) \end{pmatrix},$$

T is a finite row contraction on $\mathcal{H}$, $x \in \mathcal{H}$ *is cyclic for* T^* *and its minimal row isometric dilation,* V,

$$T_0^* := T^*\left(I - (1-\overline{\mathfrak{b}(0)})\langle x, \cdot\rangle x\right)\Big|_{\mathcal{H}_0}, \quad \mathcal{H}_0 := \bigvee_{\omega \neq \emptyset} T^{*\omega}x$$

$$\textit{and} \quad \mathfrak{b}(0) = \frac{\|x\|^2 + it - 1}{\|x\|^2 + it + 1}, \quad t \in \mathbb{R}.$$

$\mathfrak{b}$ *is inner if and only if T is also a row co-isometry.*

An example

$$T_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad T_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

T is an irreducible row co-isometry, every $x \in \mathbb{C}^2$ is cyclic for V_T and T^*.

Consider $x = \mathbf{e}_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Then,

$$T^*_{0;1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0_2, \quad T^*_{0;2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

$$\begin{aligned} \mathfrak{b}_{T,x}(Z) &= \begin{pmatrix} I, & 0 \end{pmatrix} \begin{pmatrix} I & Z_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 \\ Z_1 \end{pmatrix} \\ &= Z_2 Z_1. \end{aligned}$$

This is clearly inner. Taking $x = \mathbf{e}_2$ gives $Z_1 Z_2$.

$x = \sqrt{2}^{-1} \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. In this case, $(I - Z \otimes T_0^*)^{-1}$

$$= \begin{pmatrix} S^{-1} & S^{-1}\frac{Z_2}{2}\left(I + \frac{Z_1}{2}\right)^{-1} \\ \left(I + \frac{Z_1}{2}\right)^{-1}\frac{Z_1}{2}S^{-1} & \left(I + \frac{Z_1}{2}\right)^{-1} + \frac{1}{4}\left(I + \frac{Z_1}{2}\right)^{-1} Z_1 S^{-1} Z_2 \left(I + \frac{Z_1}{2}\right)^{-1} \end{pmatrix},$$

where

$$S := I - \frac{1}{4} Z_2 \left(I + \frac{Z_1}{2}\right)^{-1} Z_1.$$

$$\Rightarrow \mathfrak{b}_{T,x}(Z) = \frac{1}{2}\begin{pmatrix} I, & I \end{pmatrix} (I - Z \otimes T_0^*)^{-1} \begin{pmatrix} Z_2 \\ Z_1 \end{pmatrix}.$$

This must be an NC rational inner.

Mutual singularity of Clark measures

Given any $\alpha \in \partial\mathbb{D}$, let $\mu_\alpha := \mu_{b\overline{\alpha}}$ NC Clark measures, $b \in [H^\infty]_1$.

Theorem (Aronszajn–Donoghue)

The singular parts of the Clark family $\{\mu_\alpha\}$ are mutually singular.

Let $U_\alpha := M_\zeta$ on $L^2(\mu_{\alpha;s}) = H^2(\mu_{\alpha;s})$ unitary. $\mu_{\alpha;s} \perp \mu_{\beta;s} \Leftrightarrow U_\alpha \perp U_\beta$, *i.e.* have no unitarily equivalent restrictions to reducing subspaces.

Theorem (NC rational Aronszajn–Donoghue, M.–Shamovich)

Let $\mathfrak{b} \in [\mathbb{H}^\infty_d]_1$ be NC rational. Then for any $\alpha \neq \beta \in \partial\mathbb{D}$, $\Pi_{\mu_{\alpha;s}} \perp \Pi_{\mu_{\beta;s}}$.

Thank you!

Decomposition of row isometries

$\Pi : \mathcal{H} \otimes \mathbb{C}^d \to \mathcal{H}$ row isometry, $\mathscr{S}(\Pi) := \mathrm{Alg}\{I, \Pi\}^{-WOT}$ *free semigroup algebra*.

Kennedy–Lebesgue–von Neumann–Wold decomposition

$$\Pi = \Pi_L \oplus \Pi_{C-L} \oplus \Pi_{dil} \oplus \Pi_{vN}.$$

Each of $\Pi_{C-L}, \Pi_{dil}, \Pi_{vN}$ is Cuntz (surjective).

- Π_L is *pure*, *i.e.* unitarily equivalent to $L \otimes I$.
- $\mathscr{S}(\Pi_{C-L}) \simeq \mathscr{S}(L)$ *Cuntz type*$-L$.
- $\mathscr{S}(\Pi_{vN})$ is self–adjoint *von Neumann type*.
- Π_{dil} no direct summand of previous types.

$$\Pi_{dil} \simeq \begin{pmatrix} L \otimes I & * \\ & T \end{pmatrix}; \qquad T \text{ row co-isometry.}$$

Theorem (M.)

If $\mathfrak{b} \in [\mathbb{H}_d^\infty]_1$ is NC rational with finitely–correlated NC Clark measure $\mu = \mu_{\mathfrak{b}}$ then

$$\Pi_\mu = \underbrace{\Pi_{\mu;L}}_{\simeq L} \oplus \Pi_{\mu;dil}.$$

NC reproducing kernel Hilbert space

Let $\Omega = \bigsqcup_{n=1}^{\infty} \Omega_n \subseteq \mathbb{C}^d_{\mathbb{N}}$ be an NC set (closed under $\oplus$), $\Omega_n = \Omega \cap \mathbb{C}^d_n$.

NC-RKHS

A Hilbert space, $\mathcal{H}$, of NC functions on Ω is an NC reproducing kernel Hilbert space if for any $Z, y, v \in \Omega_n \times \mathbb{C}^n \times \mathbb{C}^n$,

$$f \overset{\ell_{Z,y,v}}{\mapsto} y^* f(Z) v,$$

is bounded on $\mathcal{H}$.

$$\ell_{Z,y,v}(f) = \langle K\{Z,y,v\}, f \rangle_{\mathbb{H}^2}, \quad K\{Z,y,v\} \in \mathcal{H}.$$

NC Szegö kernel

Given $Z \in \mathbb{B}^d_n, W \in \mathbb{B}^d_m$ and $P \in \mathbb{C}^{n\times m}$,

$$\mathrm{Ad}_{Z,W^*}[P] := Z_1 P W_1^* + \cdots Z_d P W_d^*.$$

$\mathbb{H}^2_d = \mathcal{H}_{nc}(K)$,

$$\begin{aligned} K(Z,W)[P] &= \left(\mathrm{id}_{n\times m} - \mathrm{Ad}_{Z,W^*}\right)^{-1} \circ P \\ &= \sum_{k=0}^{\infty} \mathrm{Ad}^{(k)}_{Z,W^*}[P], \\ &= \sum Z^\omega P W^{*\omega}. \end{aligned}$$

Compare:

$k(z,w) := \dfrac{1}{1 - z\overline{w}}$, $z, w \in \mathbb{D} = (\mathbb{C})_1$. Szegö kernel for $H^2 = \mathcal{H}(k)$.

NC reproducing kernel

Completely positive NC kernel

For $Z \in \Omega_n$, $W \in \Omega_m$, $K(Z,W)[\cdot] : \mathbb{C}^{n\times m} \to \mathbb{C}^{n\times m}$ completely bounded,

$y^* K(Z,W)[vu^*]x := \langle K\{Z,y,v\}, K\{W,x,u\}\rangle_{\mathcal{H}}; \quad y,v \in \mathbb{C}^n,\ x,u \in \mathbb{C}^m.$

$K(Z,Z)[\cdot] : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is completely positive. $\mathcal{H} =: \mathcal{H}_{nc}(K)$.

Theorem (Ball–Marx–Vinnikov)

CPNC kernels $K \leftrightarrow$ NC-RKHS $\mathcal{H}_{nc}(K)$.

Fornasini–Marchesini realizations

Any NC rational $\mathfrak{r}$, $0 \in \operatorname{Dom}\mathfrak{r}$, has an *FM realization* (A, B, C, D), $A \in \mathbb{C}^d_N$, $B \in \mathbb{C}^N \otimes \mathbb{C}^d$, $C \in \mathbb{C}^{1\times N}$, and $D = \mathfrak{r}(0) \in \mathbb{C}$: Given $Z \in \mathbb{C}^d_n$,

$$\mathfrak{r}(Z) = D + C(\underbrace{I_n \otimes I_N - \sum Z_j \otimes A_j}_{=L_A(Z)})^{-1} \sum Z_k \otimes B_k.$$

Minimal FM realizations

An FM realization is *minimal* if N is as small as possible. Given minimal (A, b, c) for $\mathfrak{r}$,

$$A' := A|_{\bigvee_{\omega\neq\emptyset} A^\omega c}, \quad B' := \begin{pmatrix} A_1 c \\ \vdots \\ A_d c \end{pmatrix}, \quad C' := b^* \quad \text{and} \quad D := \mathfrak{r}(0),$$

is a minimal FM realization.

Weak purity condition $\Rightarrow$ inner ($d=1$)

Given $b \in [H^\infty]_1$, $X := S^*|_{\mathscr{H}(b)}$, b is extreme if and only if

$$X^*X = I - K_0^b (K_0^b)^*, \qquad K_0^b = \text{point evaluation at } 0.$$

Equivalently $\|S^*b\|^2_{\mathscr{H}(b)} = 1 - |b(0)|^2$.
Weak purity: $\|X^n S^* b\|^2_{\mathscr{H}(b)} \to 0$

$$\begin{aligned}
\|X^2 b\|^2 &= \langle S^*b, X^*XS^*b\rangle_b \\
&= \langle S^*b, S^*b\rangle_b - |\langle K_0^b, S^*b\rangle|^2 = 1 - |\hat{b}_1|^2 - |b(0)|^2 \\
\ldots\|X^n S^* b\|^2 &= 1 - \sum_{j=0}^{n} |\hat{b}_j|^2 \to 0 \quad \Rightarrow \|b\|_{H^2} = 1.
\end{aligned}$$

Fact: A contractive $b \in H^\infty$ has unit H^2−norm if and only if it is inner (Davidson–Pitts).