Henkin and analytic functionals on C*-algebras

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This talk is based on joint work with Edward Timko.

Functional calculus

$\mathcal{H}$ separable Hilbert space, $T \in B(\mathcal{H})$ contraction
von Neumann's inequality: the map

$$f \mapsto f(T), \quad f \in \mathrm{A}(\mathbb{D})$$

is completely contractive

Lemma

The following conditions are equivalent.

1. *The functional calculus extends to a weak-$*$ continuous homomorphism $H^\infty(\mathbb{D}) \to B(\mathcal{H})$. (Here, we consider $\mathrm{A}(\mathbb{D}) \subset \mathrm{C}(\mathbb{T}) \subset L^\infty(\mathbb{T}, \lambda)$ and $H^\infty(\mathbb{D}) \subset L^\infty(\mathbb{T}, \lambda)$.)*
2. *If (f_n) is a bounded sequence in $\mathrm{A}(\mathbb{D})$ converging pointwise to 0 on $\mathbb{D}$, then*

$$\lim_{n\to\infty} \langle f_n(T)\xi, \eta\rangle = 0, \quad \xi, \eta \in \mathcal{H}.$$

Sz.-Nagy–Foias: this is automatically true if T is completely non-unitary.

Writing $T = T_{cnu} \oplus U$ reduces to a problem about measures on the unit circle.

Henkin and analytic measures for $\mathrm{A}(\mathbb{D})$

μ regular Borel measure on $\mathbb{T}$

Definition

- μ is analytic if $\int_{\mathbb{T}} f d\mu = 0$ for every $f \in \mathrm{A}(\mathbb{D})$.
- μ is Henkin for $\mathrm{A}(\mathbb{D})$ if whenever (f_n) is a bounded sequence in $\mathrm{A}(\mathbb{D})$ converging pointwise to 0 on $\mathbb{D}$, we have $\lim_{n\to\infty} \int_{\mathbb{T}} f_n d\mu = 0$.

Theorem (Henkin 1968, Valskii 1971)

The following assertions hold.

- *A measure is Henkin if and only if it is absolutely continuous with respect to some analytic measure. (Henkin 1968, Valskii 1971)*
- *A measure is Henkin if and only if it is absolutely continuous with respect to arc length. (the F.&M. Riesz miracle)*

Corollary

A contraction on a separable Hilbert space admits a weak-$*$ continuous $H^\infty(\mathbb{D})$-functional calculus if and only if the spectral measure of its unitary part is absolutely continuous with respect to arc length.

Multivariate versions

$d \geq 2$, $\mathbb{B}_d \subset \mathbb{C}^d$ open unit ball, $\mathbb{S}_d \subset \mathbb{C}^d$ unit sphere
σ_d surface measure on $\mathbb{S}_d$
$\mathrm{A}(\mathbb{B}_d)$ ball algebra
$\mathrm{A}(\mathbb{B}_d) \subset \mathrm{C}(\mathbb{S}_d) \subset L^\infty(\mathbb{S}_d, \sigma_d)$ and $H^\infty(\mathbb{B}_d) \subset L^\infty(\mathbb{S}_d, \sigma_d)$

Definition

Let μ be a regular Borel measure on $\mathbb{S}_d$.

- μ is analytic if $\int_{\mathbb{S}_d} f d\mu = 0$ for every $f \in \mathrm{A}(\mathbb{B}_d)$.
- μ is Henkin for $\mathrm{A}(\mathbb{B}_d)$ if whenever (f_n) is a bounded sequence in $\mathrm{A}(\mathbb{B}_d)$ converging pointwise to 0 on $\mathbb{B}_d$, we have $\lim_{n\to\infty} \int_{\mathbb{S}_d} f_n d\mu = 0$.

Henkin measures for $\mathrm{A}(\mathbb{B}_d)$

Theorem (Valskii 1971)

The measure μ is Henkin if and only if there is $g \in L^1(\mathbb{S}_d, \sigma_d)$ and an analytic measure ν on $\mathbb{S}_d$ such that $\mu = \nu + g\sigma_d$.

Theorem (Henkin 1968)

The set of Henkin measures for $\mathrm{A}(\mathbb{B}_d)$ is a band: if μ is Henkin and $\nu \ll \mu$, then ν is Henkin.

Corollary

A measure is Henkin for $\mathrm{A}(\mathbb{B}_d)$ if and only if it is absolutely continuous with respect to a convex combination of total variations of analytic measures.

Proof.

($\Leftarrow$): Use Henkin's theorem.
($\Rightarrow$): Using Valskii's theorem, it suffices to deal with σ_d. The measure $z_k\sigma_d$ is analytic for every $1 \leq k \leq d$, and σ_d is mutually absolutely continuous with $\sum_{k=1}^{d} \frac{1}{d}|z_k|\sigma_d$. □

No miracle here?

Representing measures

$\mathscr{R}_0$ – set of states μ on $C(\mathbb{S}_d)$ such that

$$\int_{\mathbb{S}_d} f d\mu = f(0), \quad f \in A(\mathbb{B}_d).$$

Note: it follows from the F. & M. Riesz theorem that $\mathscr{R}_0 = \{\lambda\}$ when $d = 1$.

$AC(\mathscr{R}_0)$ – set of $\mu \in C(\mathbb{S}_d)^*$ that are absolutely continuous with respect to some measure in $\mathscr{R}_0$

Theorem (Forelli 1963, Cole–Range 1972)

The set of Henkin measures for $A(\mathbb{B}_d)$ *is* $AC(\mathscr{R}_0)$.

Back to the functional calculus: multivariate operator theory

$\mathcal{H}$ separable Hilbert space, $T_1, \ldots, T_d \in B(\mathcal{H})$ commuting such that $\sum_{j=1}^{d} T_j T_j^* \leq I$

$\mathcal{M}_d$ – multiplier algebra of the Drury–Arveson space on $\mathbb{B}_d$ ($\approx H^\infty(\mathbb{B}_d)$)
$\mathcal{A}_d \subset \mathcal{M}_d$ norm closure of the polynomials ($\approx \mathrm{A}(\mathbb{B}_d)$)
We always have

$$\max_{z \in \overline{\mathbb{B}_d}} |f(z)| \leq \|f\|, \quad f \in \mathcal{A}_d.$$

Caution!

The inequality is strict in general. $\mathcal{A}_d$ is NOT a uniform algebra.

What is the replacement for $\mathrm{C}(\mathbb{S}_d)$?
$\mathfrak{T}_d = \mathrm{C}^*(\mathcal{A}_d) \subset B(H_d^2)$
$\mathfrak{K} \subset \mathfrak{T}_d$ and $\mathfrak{T}_d / \mathfrak{K} \cong \mathrm{C}(\mathbb{S}_d)$ ($\Rightarrow$ the dual space $\mathfrak{T}_d^*$ does not consist entirely of measures)

Theorem (Arveson 1998)

The map $f \mapsto f(T_1, \ldots, T_d)$ is completely contractive on $\mathcal{A}_d$.

Another problem about measures

Lemma

Let $T = (T_1, \ldots, T_d)$ be a commuting row contraction. The following conditions are equivalent.

1. *The functional calculus extends to a weak-$*$ continuous homomorphism on $\mathcal{M}_d$. (Here, we consider $\mathcal{A}_d \subset \mathcal{M}_d \subset B(H^2_d)$.)*
2. *If (f_n) is a bounded sequence in $\mathcal{A}_d$ converging pointwise to 0 on $\mathbb{B}_d$, then*

$$\lim_{n\to\infty} \langle f_n(T)\xi, \eta\rangle = 0, \quad \xi, \eta \in \mathcal{H}.$$

- Automatically true if $T = (T_1, \ldots, T_d)$ is completely non-unitary. (C.–Davidson 2016)
- There is a spherical unitary $U = (U_1, \ldots, U_d)$ such that $T = T_{cnu} \oplus U$ (Eschmeier)

Thus, this becomes a problem about measures on the sphere.

Henkin functionals for $\mathcal{A}_d$

Definition

A regular Borel measure μ on $\mathbb{S}_d$ is Henkin for $\mathcal{A}_d$ if whenever (f_n) is a bounded sequence in $\mathcal{A}_d$ converging pointwise to 0 on $\mathbb{B}_d$, we have $\lim_{n\to\infty} \int_{\mathbb{S}_d} f_n d\mu = 0$.

Definition

A functional $\varphi \in \mathfrak{T}_d^*$ is Henkin for $\mathcal{A}_d$ if whenever (f_n) is a bounded sequence in $\mathcal{A}_d$ converging pointwise to 0 on $\mathbb{B}_d$, we have $\lim_{n\to\infty} \varphi(f_n) = 0$.

Fact: Henkin measures for $\mathrm{A}(\mathbb{B}_d)$ are necessarily Henkin measures for $\mathcal{A}_d$.
Is the converse true?
Yes? (C.–Davidson 2016) No! (Hartz 2018)
In particular, this means that $\mathrm{AC}(\mathscr{R}_0)$ is strictly contained in the set of Henkin measures for $\mathcal{A}_d$.

Question

Can we give a description of the set of Henkin functionals for $\mathcal{A}_d$?

Strategy: adapt the classical approach for the ball algebra, using non-commutative measure theoretic ideas.

The general setting

$\mathfrak{W}$ von Neumann algebra, $\mathfrak{T} \subset \mathfrak{W}$ unital C*-subalgebra, $\mathcal{A} \subset \mathfrak{T}$ unital subalgebra

Definition

A functional $\varphi \in \mathcal{A}^*$ is said to be Henkin (relative to $\mathfrak{W}$) if whenever (a_i) is a bounded net in $\mathcal{A}$ converging to 0 in the weak-$*$ topology of $\mathfrak{W}$, we also have that $(\varphi(a_i))$ converges to 0. We denote the set of those functionals by $\mathrm{Hen}_{\mathfrak{W}}(\mathcal{A})$.

We are interested in the set

$$\mathscr{B} = \{\varphi \in \mathfrak{T}^* : \varphi|_{\mathcal{A}} \in \mathrm{Hen}_{\mathfrak{W}}(\mathcal{A})\}.$$

Define

$$\Delta = \overline{\mathrm{co}}\{|\psi| : \psi \in \mathcal{A}^{\perp}, \|\psi\| = 1\}.$$

Recall:

Theorem (Henkin, Valskii)

A measure $\mu \in \mathrm{C}(\mathbb{S}_d)^*$ satisfies $\mu|_{\mathrm{A}(\mathbb{B}_d)} \in \mathrm{Hen}_{L^\infty(\mathbb{S}_d, \sigma_d)}(\mathrm{A}(\mathbb{B}_d))$ if and only if there is $\nu \in \Delta$ such that $\mu \ll \nu$.

Goal

Given $\varphi \in \mathfrak{T}^*$, show that $\varphi \in \mathscr{B}$ if and only if there is $\psi \in \Delta$ such that $\varphi \ll \psi$.

Analytic triples

Let $\varphi, \psi \in \mathfrak{T}^*$. We write $\varphi \ll \psi$ if $|\varphi|(\xi^*\xi) = 0$ for every $\xi \in \mathfrak{T}^{**}$ satisfying $|\psi|(\xi^*\xi) = 0$.

Definition

The triple $\mathcal{A} \subset \mathfrak{T} \subset \mathfrak{W}$ is said to be analytic if a bounded net (a_i) in $\mathcal{A}$ converges to 0 in the weak-$*$ topology of $\mathfrak{W}$ whenever

$$\lim_i \varphi(a_i) = 0, \quad \varphi \in \mathrm{AC}(\Delta)$$

where $\Delta = \overline{\mathrm{co}}\{|\psi| : \psi \in \mathcal{A}^\perp, \|\psi\| = 1\}$.

Example

The following triples are analytic.

- $\mathrm{A}(\mathbb{B}_d) \subset \mathrm{C}(\mathbb{S}_d) \subset L^\infty(\mathbb{S}_d, \sigma_d)$
- $\mathcal{A}_d \subset \mathfrak{T}_d \subset B(H^2_d)$
- $\mathfrak{A}_d \subset \mathrm{C}^*(\mathfrak{A}_d) \subset B(\mathfrak{F}^2_d)$

The main result

$\mathfrak{W}$ be a von Neumann algebra, $\mathfrak{T} \subset \mathfrak{W}$ unital C*-subalgebra, $\mathcal{A} \subset \mathfrak{T}$ unital norm-closed subalgebra

$\Delta = \overline{\text{co}}\{|\psi| : \psi \in \mathcal{A}^\perp, \|\psi\| = 1\}$

$\mathscr{B} = \{\varphi \in \mathfrak{T}^* : \varphi|_\mathcal{A} \in \text{Hen}_\mathfrak{W}(\mathcal{A})\}$.

Theorem (C.–Timko 2021)

Then, the following statements hold.

1. *The triple $\mathcal{A} \subset \mathfrak{T} \subset \mathfrak{W}$ is analytic if and only if $\mathscr{B} \subset \text{AC}(\Delta)$.*
2. *Assume that the triple $\mathcal{A} \subset \mathfrak{T} \subset \mathfrak{W}$ is analytic. Then, $\mathscr{B} = \text{AC}(\Delta)$ if and only if $\mathscr{B}$ is a left band.*

A subset $\mathscr{B} \subset \mathfrak{T}^*$ is called a left band if given $\varphi \in \mathfrak{T}^*$ and $\beta \in \mathscr{B}$ with $\varphi \ll \beta$, we must necessarily have that $\varphi \in \mathscr{B}$.

Example

- For the triple $\text{A}(\mathbb{B}_d) \subset \text{C}(\mathbb{S}_d) \subset L^\infty(\mathbb{S}_d, \sigma_d)$, the set $\mathscr{B}$ is a left band.
- For the triple $\mathcal{A}_d \subset \mathfrak{T}_d \subset B(H^2_d)$, the set $\mathscr{B}$ is a left band.
- For the triple $\mathfrak{A}_d \subset \text{C}^*(\mathfrak{A}_d) \subset B(\mathfrak{F}^2_d)$, the set $\mathscr{B}$ is not a left band. (C.–Martin–Timko 2021)

A familiar question on the Drury–Arveson space

Corollary

For $\mathcal{A}_d \subset \mathfrak{T}_d = \mathrm{C}^*(\mathcal{A}_d) \subset B(H^2_d)$, *we have* $\mathscr{B} = \mathrm{AC}(\Delta)$.

$\mathscr{R}_0$ – set of states ψ on $\mathfrak{T}_d$ such that $\psi(f) = f(0), \quad f \in \mathcal{A}_d$.

Question

Do we have $\mathscr{B} = \mathrm{AC}(\mathscr{R}_0)$*?*

Yes for $d = 1$. $d \geq 2$?

Theorem (Hartz 2018)

There exists a regular Borel probability measure μ *on* $\mathbb{S}_d$ *which is Henkin for* $\mathcal{A}_d$ *yet singular with respect to all representing measures for the origin.*

$\tau_\mu \in \mathscr{B} \setminus \mathrm{AC}(\mathscr{R}_0)$?

Thank you!