Backward shifts have gone Nuclear!

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Discrete and Continuous Semigroups of Composition Operators
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Talk Map

1. Notation and Preliminaries
2. A Factorization Theorem
3. Hypercyclic Operators
4. The Operator Ideal $\mathcal{BN}_{(w,p,q)}$
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6. Some Consequences

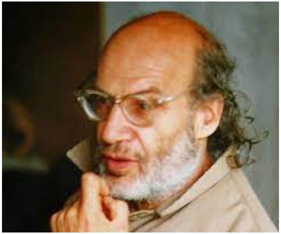
Notation

1. $\mathcal{L}(X, Y)$ is the normed vector space of all continuous operators from X to Y
2. $\mathcal{K}(X, Y)$ is the collection of all compact operators from X to Y.
3. U_X denotes the open unit ball in X,
4. Let X^* be the dual space of X, $a \in X^*$ and $y \in Y$. Define

$$a \otimes y \in \mathcal{L}(X, Y) \quad \text{as} \quad (a \otimes y)(x) := a(x)y$$

5. p' the conjugate of $p \geq 1$, i.e. $1/p + 1/p' = 1$.
6. $\ell^p(I)$, denotes the *Banach space of all absolutely p-summable scalar families* $x = (\epsilon_i)_{i \in I}$.

Alexander Grothendieck, 1928-2014



"Résumé de la théorie métrique des produits tensoriels topologiques "

Relatively Compact Sets (Grothendieck)

A subset K of Banach space X is *relatively compact* if and only if there exists a sequence $\{x_n\}$ in K such that

$$K \subseteq \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : \ |\alpha_n| \leq 1 \right\} \quad \text{and} \quad \lim_{n\to\infty} ||x_n|| = 0$$

$T \in \mathcal{L}(X, Y)$ is said to be *compact* if every bounded subset of X is mapped to a relatively compact subset of Y that is

$$T(U_X) \subseteq \left\{ \sum_{n=1}^{\infty} \alpha_n y_n : \ |\alpha_n| \leq 1 \right\} \quad \text{and} \quad \lim_{n\to\infty} ||y_n|| = 0$$

Weakly p-summable Sequences

For $1 \leq p < \infty$, the space of all *weakly p-summable sequences* in a Banach space X is defined as

$$l_p^w(X) = \{(x_i) \in X^{\mathbb{N}} : (a(x_i))_i \in l_p, \forall a \in X^*\},$$

with norm

$$\|(x_i)\|_p^w := \sup\{\|(a(x_i))_i\|_p : a \in X^*, \|a\| \leq 1\}.$$

i.e., $(x_i) \in l_p^w(X)$ if and only if $(a(x_i)) \in l^p \quad \forall a \in X^*$.

Weak* p-summable Sequences

The space of all *weak* p-summable sequences* in X^* is defined as

$$l_p^{w*}(X^*) = \{(a_i) \in (X^*)^{\mathbb{N}} : (a_i(x))_i \in l_p, \forall x \in X\},$$

with norm

$$\|(a_i)\|_p^{w^*} := \sup\{\|(a_i(x))_i\|_p : x \in X, \|x\| \leq 1\}.$$

i.e., $(a_i) \in l_p^{w^*}(X^*)$ if and only if $(a_i(x)) \in l^p \quad \forall x \in X$.

Isometric identification

Let $1/p + 1/p' = 1$

- $\ell_p^w(X)$, $1 < p < \infty$ is isometrically identified with $\mathcal{L}(\ell_{p'}, X)$.

  For $(x_i) \in \ell_p^w X$, define $E_{(x_i)} : \ell_{p'} \to X$ by $(\lambda_i) \mapsto \sum_{i=1}^{\infty} \lambda_i x_i$

- $\ell_p^{w^*}(X^*)$, $1 \leq p < \infty$ is isometrically identified with $\mathcal{L}(X, \ell_p)$,

  For $(a_i^*) \in \ell_p^{w^*}(X^*)$, define $F_{a_i^*} : X \to \ell_p$ by $x \mapsto (a_i^*(x))_i$.

Tosun Terzioğlu, 1942-2016

A characterization of compact linear mappings, Arch. Math.**22** (1971), 76 – 78.

Factorization by T. Terzioğlu

Theorem

Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator between Banach spaces X and Y. Then the following are equivalent:

1. $T \in \mathcal{K}(X, Y)$,
2. *There exists a norm-null sequence (x_n^*) in X^* such that*

$$||Tx|| \leq \sup_n |\langle x, x_n^* \rangle| \quad \forall x \in X,$$

3. *For some closed subspace Z of c_0 there are compact operators $P \in \mathcal{K}(X, Z)$ and $Q \in \mathcal{K}(Z, Y)$ such that $T = Q \circ P$.*

Hypercyclic Operator

Definition

A linear bounded linear operator T acting on a separable infinite dimensional Banach space X is called hypercyclic if there exists $x \in X$ such that $\{T^n(x) : n \geq 0\}$ is dense in X.

Question: How to prove an operator is hypercyclic?

Typical Example - Hypercyclic Operators

S. Rolewicz, 1932 – 2015

Let $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the backward shift operator defined by $B(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Then

$$\lambda B : (x_n)_n \to \lambda(x_{n+1})_n$$

is hypercyclic for any scalar $|\lambda| > 1$

Basic Properties

1. There are no hypercyclic operators on finite dimensional space $X \neq \{0\}$.
2. No compact operator on a complex (or real) Banach space $X \neq \{0\}$ can be hypercyclic.
3. Any separable infinite-dimensional Banach space admits a bounded hypercyclic operator
4. Any operator of the form $\lambda I+$(backward shift) is hypercyclic.

Hypercyclic Operators are not Compact

Theorem

Let X be a complex Banach space, and let $T \in \mathcal{L}(X)$ be hypercyclic. Then every connected component of the spectrum of T intersect the unit circle.

No compact operator on a complex Banach space $X \neq \{0\}$ can be hypercyclic.

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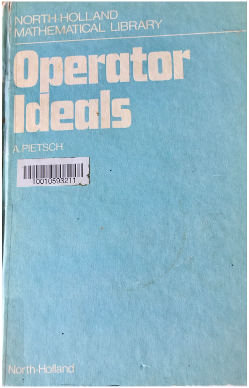
Hypercyclic Operators are not Compact However

We are interested in studying operators satisfying

$$T(U_X) \subseteq \left\{ \sum_n \alpha_n y_n : (\alpha_n) \in U_{\ell^{q'}} \right\} \tag{1}$$

where (y_n) is a weakly q-summable sequence in X with $1/q + 1/q' = 1$. These operators can be hypercyclic. Indeed, weighted backward shifts are examples of operators satisfying (1).

A. Pietsch, Operator Ideals



Ideal of Operators

An ideal of operators between Banach spaces is an assignment $\mathcal{U}$ which associates with every pair (X, Y) of Banach spaces a subset $\mathcal{U}(X, Y)$ of $\mathcal{L}(X, Y)$ such that following conditions are satisfied for arbitrary Banach spaces X_0, X, Y, Y_0:

1. $x^* \otimes y : X \to Y : x \mapsto x^*(x)y$ belongs to $\mathcal{U}(X, Y)$ $\forall x^* \in X^*, \forall y \in Y$,
2. if $T_1, T_2 \in \mathcal{U}(X, Y)$ then $T_1 + T_2 \in \mathcal{U}(X, Y)$
3. if $A \in \mathcal{L}(X_0, X)$, $T \in \mathcal{U}(X, Y)$ and $B \in \mathcal{L} \in (Y, Y_0)$, then $BTA \in \mathcal{U}(X_0, Y_0)$.

$$X_0 \xrightarrow{A} X \xrightarrow{T} Y \xrightarrow{B} Y_0$$

Ideal Norm

A rule α is a ideal norm if $(\mathcal{U}(X,Y), \alpha(\cdot))$ is a normed space and for which

1. $$\alpha(x^* \otimes y) = ||x^*||||y||$$
   for all $x^* \in X^*$ and for all $y \in Y$,
2. $$\alpha(BTA) \leq ||B||\,\alpha(T)\,||A||$$
   for all operators T, A and B.

Each such pair $(\mathcal{U}, \alpha)$ is called Banach operator ideal.

Examples

1. $\mathcal{K}(X, Y)$ Ideals of compact operators
2. $\mathcal{N}(X, Y)$ Ideals of nuclear operators

$T : X \to Y$ is nuclear if $T = \sum_{i=1}^{\infty} a_i \otimes y_i$ with $a_i \in X^*$ and $y_i \in Y$ such that $\sum_{i=1}^{\infty} ||a_i||.||y_i|| < \infty$.

$$||T||_{\mathcal{N}} = \inf \sum_{i=1}^{\infty} ||a_i||.||y_i||$$

$(\mathcal{N}, ||T||_{\mathcal{N}})$ is a normed operator ideal.

Factorization of Nuclear Operators-Grothendieck

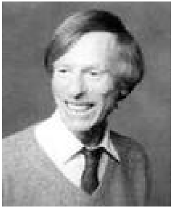
$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ A\downarrow & & \uparrow B \\ \ell_\infty & \xrightarrow{D_t} & \ell_1 \end{array}$$

Where the diagonal operator $D_t : (\xi_k) \mapsto (\tau_k \xi_k)$ is induced by the sequence $(\tau_k) \in \ell_1$.

Testing Ground: Weighted Shifts

1. He, Huang, Yu showed that if X is a Banach space in which $(e_n)_n$ is a basis and if a weighted backward shift $B_w : X \to X$ admits an orbit with a nonzero limit point, then B_w is hypercyclic.
2. H.N. Salas characterizes hypercyclic forward weighted shifts in terms of their weight sequences.

Testing Ground: Weighted Shifts



1927 -1989

A. L. Shields uses weighted shifts in analytic function theory.

Question

Can one use backward shift to obtain factorization of hypercyclic operators?

The (w, p, q)-backward Nuclear Operators

Definition

Let $1 \leq p, q \leq \infty$ be such that $1/p + 1/q \leq 1$. An operator $T \in \mathcal{L}(X, Y)$ is called *(w, p, q)-backward nuclear* if

$$T = \sum_{i=2}^{\infty} w_i a_i \otimes y_{i-1}$$

with $(w_i) \in \ell^{\infty}$, $(a_i) \in \ell_{q'}^{w^*}(X^*)$ and $(y_i) \in \ell_{p'}^{w}(Y)$.

Norm of (w, p, q)-backward Nuclear Operators

Its norm is defined by

$$\|T\|_{(w,p,q)} = \inf \|w\|_\infty \|(a_i)\|_{q'}^{w^*} \|(y_i)\|_{p'}^{w},$$

where the infimum is taken over all (w, p, q)-backward nuclear representations described above.

Notation

$$\mathcal{BN}_{(w,p,q)}$$

will denote the class of all (w, p, q)-backward nuclear operators.

Example

Example

Weighted backward shifts on ℓ^p are examples of (w, p, p')-backward nuclear operators, as they are defined as

$$B_w e_n := w_n e_{n-1}, \;\; n \geq 1 \quad \text{with} \quad B_w e_0 := 0$$

where $(e_n)_{n\in\mathbb{N}}$ denotes the canonical basis of ℓ^p and B_w is the unilateral weighted backward shift operator.

(w, p, q) - representation of B_w

Example

$$B_w = \sum_{i=2}^{\infty} w_i e_i^* \otimes e_{i-1} \in \mathcal{BN}_{(w,p,q)}(\ell^{q'}, \ell^p)$$

Is it an Operator Ideal?

Theorem (A., Puig)

Let $1/s = 1/p' + 1/q'$, then $(\mathcal{BN}_{(w,p,q)}, \|\cdot\|_{(w,p,q)})$ is an s-Banach operator ideal.

Factorization property of $\mathcal{BN}_{(w,p,q)}$.

Theorem (A., Puig)

Let $T \in \mathcal{L}(X,Y)$ and $B_w \in \mathcal{L}(\ell^{q'}, \ell^p)$ be a unilateral backward weighted shift. The following are equivalent:

1. $T \in \mathcal{BN}_{(w,p,q)}(X,Y)$,
2. *there exists* $S \in \mathcal{L}(X, \ell^{q'})$ *and* $R \in \mathcal{L}(\ell^p, Y)$ *such that*

$$T = RB_wS.$$

Moreover, $\|T\|_{(w,p,q)} = \inf \|R\| \cdot \|B_w\| \cdot \|S\|$ *where the infimum is taken over all possible factorizations.*

Factorization Property of $\mathcal{BN}_{(w,p,q)}$.

$T \in \mathcal{BN}_{(w,p,q)}$ if and only if T admits a factorization:

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ S \downarrow & & \uparrow R \\ \ell^{q'} & \xrightarrow{B_w} & \ell^{p} \end{array}$$

$Sx = (a_i(x))_i$ for any $x \in X$ and $R((\nu_i)) = \sum_{i=1}^{\infty} \nu_i y_i$ for any $(\nu_i)_i \in \ell_p$.

Idea of the proof

Let $T \in \mathcal{BN}_{(w,p,q)}(X, Y)$ and $\varepsilon > 0$. Consider a backward nuclear representation $T = \sum_{i=2}^{\infty} w_i a_i \otimes y_{i-1}$ such that

$$\|w\|_\infty \|(a_i)\|_{q'}^{w*} \|(y_i)\|_{p'}^{w} \leq (1+\varepsilon)\|T\|_{(w,p,q)}$$

Consider the operators $S \in \mathcal{L}(X, \ell_{q'})$ and $R \in \mathcal{L}(\ell_p, Y)$ defined as $Sx = (a_i(x))_i$ for any $x \in X$ and $R((\nu_i)) = \sum_{i=1}^{\infty} \nu_i y_i$ for any $(\nu_i)_i \in \ell_p$. It is well-known that

$$\|S\| = \|(a_i)\|_{q'}^{w*} \qquad \text{and} \qquad \|R\| = \|(y_i)\|_{p'}^{w}$$

Hence, $\|B_w\|_\infty \|S\| \|R\| \leq (1+\varepsilon)\|T\|_{(w,p,q)}$ and $RB_wS = T$.

Idea of the proof

Conversely, by definition
$B_w = \sum_{i=2}^{\infty} w_i e_i^* \otimes e_{i-1} \in \mathcal{BN}_{(w,p,q)}(\ell^{q'}, \ell^p)$ with

$$\|B_w\|_{(w,p,q)} \leq \|w\|_\infty = \|B_w\|$$

where $e_i^*(e_i) = 1$ and $e_i^*(e_j) = 0$ for all $i \neq j$.
Since $\mathcal{BN}_{(w,p,q)}$ is an operator ideal, we have that

$$T = RB_wS = \sum_{i=2}^{\infty} w_i S^* e_i^* \otimes Re_{i-1} \in \mathcal{BN}_{(w,p,q)}(X, Y)$$

and

$$\|T\|_{(w,p,q)} \leq \|R\| \|B_w\|_{(w,p,q)} \|S\| \leq \|R\| \|B_w\| \|S\|.$$

Some Consequences

Theorem (A., Puig)

Let $T \in \mathcal{L}(X,Y)$ and $B_w \in \mathcal{L}(\ell^{q'}, \ell^p)$ be a unilateral backward weighted shift. The following are equivalent:

(1) $T \in \mathcal{BN}^{sur}_{(w,p,q)}(X,Y)$,

(2) there exists $P \in \mathcal{BN}_{(w,p,q)}(\ell^{q'}, Y)$ of the form

$Px := \sum_{i=1}^{\infty} x_{i+1} w_{i+1} y_i$ *with $(y_i) \in \ell_q^w(Y)$ satisfying*

$$T(U_X) \subseteq \left\{ Px : (x_n) \in U_{\ell^{q'}} \right\}.$$

Moreover,

$$\|T\|^{sur}_{(w,p,q)} = \inf \left\{ \|P\|_{(w,p,q)} : T(U_X) \subseteq P(U_{\ell^{q'}}), P \in \mathcal{BN}_{(w,p,q)} \right\}.$$

Some Consequences

Theorem (A., Puig)

$$(\mathcal{BN}^{inj}_{(w,p,p')})^{dual} = \mathcal{BN}^{sur}_{(w,p',p)}.$$

A linear and bounded operator T acting on X is said to be *dual hypercyclic* when both T and its adjoint T^* are hypercyclic. This theorem has connections with the dual hypercyclicity of operators belonging either to the injective or the surjective hull of $\mathcal{BN}_{(w,p,q)}$.

Questions

1. How about other classes of operators? (What is so special about weighted shifts?)
2. Connection to Q-compact operators?

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Thanks


どうも
VIELEN DANK
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THANK YOU
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