Mean Lipschitz conditions and composition semigroups

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Introduction

$\mathbb{D}$=unit disc in $\mathbb{C}$,
$\mathbb{T}$=the unit circle.

For $f : \mathbb{D} \to \mathbb{C}$ analytic, the integral means on circles of radius $r < 1$ are

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p},$$

for $1 \le p < \infty$ and

$$M_\infty(r, f) = \max_{-\pi \le \theta < \pi} |f(re^{i\theta})|.$$

Both are increasing functions of r.

The Hardy space H^p consists of those f for which

$$\|f\|_p = \sup_{r<1} M_p(r,f) = \lim_{r\to 1} M_p(r,f) < \infty.$$

For $f \in H^p$ the boundary function

$$f^*(\theta) = \lim_{r\to 1} f(re^{i\theta})$$

exists a.e. on the circle, $\|f\|_p = \|f^*\|_{L^p(\mathbb{T})}$, and

$$\int_{-\pi}^{\pi} |f(re^{i\theta}) - f^*(\theta)|^p \, d\theta \to 0,$$

as $r \to 1$. Thus for the dilations $f_r(z) = f(rz)$, $\|f_r^* - f^*\|_{L^p(\mathbb{T})} \to 0$.

For $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, the analytic mean Lipschitz spaces are defined by taking the periodic extension of f^* on $\mathbb{R}$ and considering the *p-modulus of continuity*

$$\omega_p(f,t) = \sup_{\theta,\ |h| \leq t} \|f^*(\theta + h) - f^*(\theta)\|_{L^p(\mathbb{T})}.$$

Then the spaces are

$$\Lambda\,(p,\alpha) =: \{f \in H^p : \omega_p(f,t) = O(t^\alpha)\}$$

and ("little oh" analogue),

$$\lambda\,(p,\alpha) =: \{f \in H^p : \omega_p(f,t) = o(t^\alpha),\ t \to 0\}$$

In other words

$$f \in \Lambda\,(p,\alpha) \Longleftrightarrow \|f^*(\theta+t)-f^*(\theta)\|_{L^p(\mathbb{T})} \le C|t|^\alpha,$$

for a constant C. This is clearly equivalent to

$$\|f(e^{it}z)-f(z)\|_p \le C|t|^\alpha.$$

These spaces were studied already in the work of Hardy and Littlewood (1920's) in connection with the convergence and summability of Fourier series.

Some basic properties:

• For $p = \infty$, $\Lambda(\infty, \alpha)$ is the classical space Λ_α of functions $f : \mathbb{D} \to \mathbb{C}$ analytic with

$$|f(z) - f(w)| \leq C|z - w|^\alpha, \;\; z, w \in \mathbb{D}.$$

• The norm

$$\|f\|_{p,\alpha} = |f(0)| + \sup_{-\pi \leq t < \pi} \frac{\|f^*(\theta + t) - f^*(\theta)\|_p}{|t|^\alpha}$$

makes $\Lambda(p, \alpha)$ into a nonseparable Banach space.

• Functions in $\Lambda(p, \alpha)$ have a certain degree of "smoothness".

• $\Lambda(p, \alpha)$ decreases in size as either p or α increases (with the other index kept fixed).

• For $\alpha > 1/p$, every $f \in \Lambda(p, \alpha)$ is continuous on $\overline{\mathbb{D}}$. The value $\alpha = 1/p$ is the borderline for continuity on the closed disc. Indeed,

• If $1 < p < q < \infty$ then

$$\log(\frac{1}{1-z}) \in \Lambda(p, 1/p) \subset \Lambda(q, 1/q) \subset BMOA.$$

• $f \in \Lambda(1, 1)$ if and only if $f' \in H^1$.

The theorem of Hardy-Littlewood

Theorem [H-L]. If $1 \leq p < \infty$, $0 < \alpha \leq 1$, then the following are equivalent for $f \in H^p$:

(a) $f \in \Lambda\,(p, \alpha)$,
(b) $M_p(r, f') = \mathrm{O}\left((1-r)^{\alpha-1}\right), \quad r \to 1.$

Note that condition (b) can be restated as

$$\|(f')_r\|_p = \mathrm{O}\left((1-r)^{\alpha-1}\right),$$

and the theorem of H-L may be interpreted as saying:

Smoothness of the boundary function f^* can be detected from inside the disc by examining the dilations of the derivative f'.

Extension for Abel means

The next theorem says that we can detect smoothness of f^* by using the dilations of f itself.

Theorem. Let p and α be as before. Then for $f \in H^p$ the following are equivalent

(b) $M_p(r, f') = \mathrm{O}\left((1-r)^{\alpha-1}\right), \quad r \to 1,$
(c) $\|f_r - f\|_p = \mathrm{O}((1-r)^{\alpha}), \quad r \to 1.$

Remark:
In contrast to the theorem of H-L, this result is not mentioned in works on the topic. Some special cases (in one direction) can be found sporadically in some works, but we could not locate the full theorem in "western" articles.

It turned out that the full theorem was proved in a highly technical paper by E. A. Storozhenko (Math. Sbornik, 1982), while part of it was proved in her thesis (1978). She attributes some special cases to other Soviet mathematicians, almost all articles being in Russian. In her work the theorem is stated in the form,

$$\|f_r - f\|_p \asymp \omega_p(f, 1-r), \quad r \to 1.$$

We will sketch an elementary proof on Hardy spaces and proceed to examine Lipschitz conditions on Bergman spaces.

Sketch of our proof
(Joint work with P. Galanopoulos and G. Stylogiannis).

$\underline{(b) \Rightarrow (c)}$. Suppose that (b) holds, i.e.

$$M_p(r, f') = \mathrm{O}\left((1-r)^{\alpha-1}\right), \quad r \to 1.$$

Start with the identity

$$(\star) \quad f(z) - f(rz) = \int_r^1 z f'(sz)\, ds.$$

Taking integral means on $|z| = u \in (0, 1)$ and using Minkowski's inequality we have

$$M_p(u, f - f_r) \leq u \int_r^1 M_p(su, f')\, ds.$$

Then take supremum on u,

$$\|f_r - f\|_p \leq \int_r^1 M_p(s, f')\, ds,$$

and use the assumtion to obtain

$$\|f_r - f\|_p \leq C \int_r^1 (1-s)^{\alpha-1} = \frac{C}{\alpha}(1-r)^{\alpha},$$

the desired conclusion.

<u>Conversely</u> to show that $(c) \Rightarrow (b)$, we use the modified identity,

$$(\star') \ (1-r)f'(z) = \frac{f(z) - f(rz)}{z} + \int_r^1 (f'(z) - f'(sz))\, ds.$$

For r fixed, take integral means on $|z| = u$,

$$(1-r)M_p(u, f') \leq M_p(u, \frac{f - f_r}{z}) + \int_r^1 M_p(u, \Phi'_{[s]})\, ds$$

where we have written

$$\Phi_{[s]}(z) = f(z) - \frac{1}{s} f_s(z), \;\; r \leq s < 1.$$

For the first term in the sum we have,

$$M_p(u, \frac{f - f_r}{z}) \leq \|f_r - f\|_p.$$

To estimate the second term we use the inequality

$$M_p(u, F') \leq \frac{C\|F\|_p}{1-u}, \quad u < 1,$$

valid for functions $F \in H^p$.

Apply this to $\Phi_{[s]} = f - \frac{1}{s} f_s$ and use the triangle inequality to obtain,

$$\begin{aligned}(1-u)M_p(u,\Phi'_{[s]}) &\leq C\|\Phi_{[s]}\|_p \\ &\leq \|f - f_s\| + \|f_s - \frac{1}{s} f_s\| \\ &= \|f_s - f\| + \frac{1-s}{s}\|f_s\| \\ &\leq C\|f_s - f\| + \frac{C}{r}(1-s)\|f\|,\end{aligned}$$

valid for $0 \leq u < 1$ and $r \leq s < 1$.

Now take $u = r$ and integrate on $[r, 1)$ with respect to s,

$$\int_r^1 M_p(r, \Phi'_{[s]})\, ds \le \frac{C}{1-r} \int_r^1 \|f_s - f\|\, ds + \frac{C\|f\|}{2r}(1-r).$$

This is the estimate for the second term. Put the two estimates together,

$$(1-r)M_p(r, f') \le \|f_r - f\|_p + \frac{C}{1-r}\int_r^1 \|f_s - f\|_p\, ds + \frac{C\|f\|_p}{2r}(1-r),$$

and use the assumption $\|f_r - f\|_p \le C'(1-r)^\alpha$ to conclude $M_p(r, f') = \mathrm{O}((1-r)^{\alpha-1})$, QED.

Corollary: If p and α are as before and $f \in H^p$ then the following are equivalent
(a) $f \in \lambda(p, \alpha)$,
(b) $M_p(r, f') = \mathrm{o}\left((1-r)^{\alpha-1}\right), \quad r \to 1$,
(c) $\|f_r - f\|_p = \mathrm{o}((1-r)^\alpha), \quad r \to 1$.

Other scales of approximation:

Let

$$\omega : [0, 1) \to [0, \infty), \quad \omega(0) = 0,$$

a continuous, nondecreasing function (weight).

$\Lambda(p, \omega)$ consists of $f \in H^p$ such that

$$\|f^*(\theta + t) - f^*(\theta)\|_p = \mathrm{O}\left(\omega(t)\right), \quad t \to 0.$$

($\omega(t) = t^\alpha$ corresponds to $\Lambda(p, \alpha)$.)

We need to restrict to good weights.

ω is a *Dini weight* if

$$\int_0^t \frac{\omega(s)}{s}\, ds \leq C\omega(t), \qquad 0 < t < 1,$$

and it is *admissible Dini weight* if in addition

$$\int_t^1 \frac{\omega(s)}{s^2}\, ds \leq C\frac{\omega(t)}{t}, \qquad 0 < t < 1,$$

For admissible weights the H-L theorem for $\Lambda(p,\omega)$ takes the form

Theorem (O. Blasco-G.S. de Souza, 1990). *Let* $1 \le p < \infty$, ω *admissible, and* f *analytic on* $\mathbb{D}$. *Then the following are equivalent*

(a) $f \in \Lambda\,(p, \omega)$,
(b) $M_p(r, f') = \mathrm{O}\left(\frac{\omega(1-r)}{1-r}\right), \quad r \to 1.$

Adapting the arguments of the previous proof we obtain,

Theorem. Under the same assumptions on p and ω the following are equivalent
(b) $M_p(r, f') = \mathrm{O}\left(\frac{\omega(1-r)}{1-r}\right), \quad r \to 1,$
(c) $\|f_r - f\|_p = \mathrm{O}(\omega(1-r)), \quad r \to 1.$

On Bergman spaces:

For $1 \le p < \infty$, the Bergman space A^p consists of functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dm(z) < \infty$$

where $dm(z) = \frac{1}{\pi} r d\theta dr$.

We want to study "Lipschitzness" of Bergman functions.

Unlike Hardy spaces, functions in A^p do not necessarily have boundary values on the circle, (to be used for the definition of modulus of continuity). But since for $f \in H^p$

$$\|f^*(\theta + t) - f^*(\theta)\|_{L^p(\mathbb{T})} = \|f(e^{it}z) - f(z)\|_{H^p},$$

we may use the analogous quantity

$$\|f(e^{it}z) - f(z)\|_{A^p},$$

as a measure of "smoothness" in A^p.

For f analytic on $\mathbb{D}$ write

$$A_p(r,f) = \|f_r\|_{A^p},$$

and note that

$$\begin{aligned} A_p(r,f) &= \left(\int_{\mathbb{D}} |f(rz)|^p \, dm(z)\right)^{1/p} \\ &= \left(\frac{1}{m(r\mathbb{D})} \int_{r\mathbb{D}} |f(z)|^p \, dm(z)\right)^{1/p} \end{aligned}$$

is an *area integral mean*, much the same as

$$M_p(r,f) = \|f_r\|_{H^p} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}.$$

is an arc-length integral mean.

It is easy to see that if $f \in A^p$ then

$$A_p(r, f) = \|f_r\|_{A^p} \to \|f\|_{A^p}, \quad r \to 1,$$

and an application of Lebesgue dominated convergence theorem shows

$$\|f_r - f\|_{A^p} \to 0, \quad r \to 1.$$

The following theorem relates the rate of approximation of f by $f(e^{it}z)$ in the Bergman norm to the growth of $A_p(r, f') = \|(f')_r\|_{A^p}$ and to the rate of approximation of f by f_r. That is, it is an analogue for Bergman spaces of what we know for H^p.

Theorem. *Let* $1 \le p < \infty$, $0 < \alpha \le 1$ *and* $f \in A^p$. *Then the following are equivalent*

(a) $\|f(e^{it}z) - f(z)\|_{A^p} = \mathrm{O}(|t|^\alpha)$,
(b) $A_p(r, f') = \mathrm{O}\left((1-r)^{\alpha-1}\right), \quad r \to 1$,
(c) $\|f_r - f\|_{A^p} = \mathrm{O}((1-r)^\alpha), \quad r \to 1$.

The proof follows the lines of the Hardy space case, i.e. is based on the identities ($\star$) together with

- A decomposition

$$f(e^{it}z) - f(z) = \int_z^{\delta z} + \int_{\delta z}^{\delta e^{it} z} + \int_{\delta e^{it} z}^{e^{it} z} f'$$

for a $\delta < 1$, whose value is appropriately. This kind of argument was used already by Hardy and Littlewood.

- The inequality

$$\|g(z)/z\|_{A^p} \leq C_p \|g\|_{A^p},$$

which is valid for $g \in A^p$ vanishing at 0.

- The inequality

$$A_p(u, F') \leq \frac{C_p \|F\|_{A^p}}{1-u}, \quad u < 1, \quad F \in A^p.$$

This inequality is a consequence of

$$\|F\|_{A^p}^p \asymp |F(0)|^p + \int_{\mathbb{D}} |F'(z)|^p (1-|z|^2)^p \, dm(z).$$

On the Dirichlet space.

Recall

$$f \in \mathcal{D} \Leftrightarrow f' \in A^2,$$

with $\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \|f'\|_{A^2}^2$.

Functions in $\mathcal{D}$ have boundary values because $\mathcal{D} \subset H^2$, but we continue to use the quantity

$$\|f(e^{it}z) - f(z)\|_{\mathcal{D}}$$

as a measure of smoothness since the corresponding quantity in terms of the boundary function "is not nicer".

We write

$$D(r,f) = \|f_r\|_{\mathcal{D}}.$$

Using the Bergman space result and simple arguments such as triangle inequalities, we obtain as a corollary,

Theorem. *Let* $0 < \alpha \leq 1$ *and* $f \in \mathcal{D}$. *The following are equivalent*

(a) $\|f(e^{it}z) - f(z)\|_{\mathcal{D}} = \mathrm{O}(|t|^{\alpha})$,
(b) $D(r,f') = \mathrm{O}\left((1-r)^{\alpha-1}\right), \quad r \to 1$,
(c) $\|f_r - f\|_{\mathcal{D}} = \mathrm{O}((1-r)^{\alpha}), \quad r \to 1$.

On the disc algebra $\mathcal{A}$

For $0 < \alpha \leq 1$ the classical Lipschitz space $\Lambda_\alpha(\mathbb{D})$ contains the functions such that

$$|f(z) - f(w)| \leq C|z - w|^\alpha, \quad z, w \in D,$$

For each α, $\Lambda_\alpha(\mathbb{D}) \subset \mathcal{A}$, the disc algebra.

Hardy and Littlewood had proved that the following are equivalent

(a) $f \in \Lambda_\alpha(\mathbb{D})$,
(b) $M_\infty(r, f') = \mathrm{O}((1 - r)^{\alpha-1})$,

and by taking H^∞-means in the identity $(\star)$, it is easily seen that (b) is equivalent to

(c) $\|f_r - f\|_\infty = \mathrm{O}((1 - r)^\alpha)$.

Collecting all the above:

if X is any of the spaces

$$X = H^p, A^p, \mathcal{D}, \mathcal{A},$$

then for $0 < \alpha \leq 1$ and $f \in X$ the following are equivalent

(a) $\|f(e^{it}z) - f(z)\|_X = \mathrm{O}(|t|^\alpha)$,
(b) $\|(f')_r\|_X = \mathrm{O}\left((1-r)^{\alpha-1}\right), \quad r \to 1$,
(c) $\|f_r - f\|_X = \mathrm{O}((1-r)^\alpha), \quad r \to 1$.

Favard classes

Let X be a Banach space and (T_t) a c_0 semigroup of bounded operators on X. For $0 < \alpha \leq 1$ the Favard class F_α for (T_t) is

$$F_\alpha = \{f \in X : \|T_t(f) - f\|_X = O(t^\alpha)\}.$$

and its "little oh" version

$$X_\alpha = \{f \in X : \|T_t(f) - f\|_X = o(t^\alpha) \ \ (t \to 0)\}.$$

(abstract Hölder class). These are subspaces of X used in the approximation theory of semigroups

Consider $X = H^p$ and suppose (T_t) is a semigroup of composition operators i.e.

$$T_t(f)(z) = f(\phi_t(z)),\ t \geq 0$$

where (ϕ_t) is a semigroup of analytic self-maps of $\mathbb{D}$. In this setting the results we have described previously may be interpreted as saying that for the semigroup of rotations

$$\phi_t(z) = e^{it}z$$

the resulting Favard class is

$$F_\alpha = \Lambda(p, \alpha),$$

The same is true for the semigroup of dilations

$$\phi_t(z) = e^{-t}z,$$

taking into account that $(1 - e^{-t})^\alpha \sim t^\alpha$ near 0.

Question What is the Favard class for a composition semigroup induced by a general (ϕ_t)?

Observe that for rotation and dilation semigroup the infinitesimal generator is

$$G(z) = \frac{\partial \phi_t(z)}{\partial t}|_{t=0} = iz, \text{ or } -z$$

so that the the function F of positive real part which gives the generator as $G(z) = -zF(z)$ is a constant function.

Conjecture: Initial calculations show that the Favard class for $T_t(f) = f \circ \phi_t$ on H^p may be described by looking at the rate of growth of

$$M_p(r, Gf')$$

with G the generator of (ϕ_t), at least in the case when the Denjoy-Wolff point of (ϕ_t) is inside the disc.

Several relevant questions arise, for example:

- If $\{\phi_t\}$ and $\{\psi_t\}$ are semigroups with generators G_1 and G_2 what under what conditions on G_1, G_2 they produce the same Favard class (say on Hardy spaces)?

- What are the Favard classes of semigroups with Denjoy-Wolff point on the circle $\mathbb{T}$?

- If we stay with the semigroups of rotations and dilations, on what other function spaces X do we have a description of the Favard class by the condition $\|(f')_r\|_X = \mathrm{O}\left(1-r)^{\alpha-1}\right)$?

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Thank you for your attention