Absolutely summing composition operators on Bloch spaces.

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Workshop on Discrete and Continuous Semigroups of Composition Operators

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Framework

Notations:

- $\mathbb{T} = \{z \in \mathbb{C} \,|\, |z| = 1\} = \partial\mathbb{D}$
- A normalized area measure on $\mathbb{D}$.

Theme: No surprise! We shall focus on composition operators...

Given a symbol : $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ analytic

and a space of analytic functions X on $\mathbb{D}$

The composition operator C_φ is

$$f \in X \longmapsto C_\varphi(f) = f \circ \varphi \qquad (\in X\ ?) \qquad (\in Y\ ?)$$

Composition operators

A few natural questions:

- When C_φ is bounded ?
- When C_φ is compact ?
- More generally understand the link: "Operator C_φ" $\overset{??}{\longleftrightarrow}$ "symbol φ"

  So that the aim of this area is to build a bridge (or a dictionary) between

  Operator theory and **Function theory**.

The new results will concern nuclear and absolutely summing operators on Bloch spaces...

Bloch-type spaces on the unit disk

We shall focus on Bloch type spaces

Let $\beta > 0$,

- $$\mathscr{B}^\beta = \left\{ f \in \mathscr{H}(\mathbb{D}) \middle| \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f'(z)| < \infty \right\}$$
  and
  $$\|f\|_{\mathscr{B}^\beta} = |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |f'(z)|$$

- $$\mathscr{B}_0^\beta = \left\{ f \in \mathscr{H}(\mathbb{D}) \middle| \lim_{|z|\to 1} (1-|z|^2)^\beta |f'(z)| = 0 \right\}$$
  The Little Bloch space $\mathscr{B}_0^\beta$ is a closed subspace of $\mathscr{B}^\beta$.

When $\beta = 1$, we recover classical Bloch spaces $\mathscr{B}$ and $\mathscr{B}_0$.

$\mathscr{B}^\beta$ and $\mathscr{B}_0^\beta$ are Banach spaces and $\gamma > \beta \implies \mathscr{B}_0^\beta \subset \mathscr{B}_0^\gamma$

Basic properties...

- $H^\infty = \{f \in \mathscr{H}(\mathbb{D});\ \|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < +\infty\}$

When $\beta \geq 1$: $H^\infty \subset \mathscr{B} \subset \mathscr{B}^\beta$.

In fact, it is a consequence of the Schwarz-Pick lemma:
since, for $f \in H^\infty(\mathbb{D})$ with $\|f\|_\infty \leq 1$,

$$\forall z \in \mathbb{D}, \qquad (1-|z|^2)^\beta |f'(z)| \leq (1-|z|^2)|f'(z)| \leq 1 - |f(z)|^2 \leq 1.$$

Actually $H^\infty \subsetneq \mathscr{B}^\beta$: the function $f(z) = \log(1-z)$ belongs to $\mathscr{B}$ but not to H^∞.

When $\beta < 1$: $\mathscr{B}^\beta \subset A(\mathbb{D}) \subset H^\infty$.

Basic properties...

Duality

$(\mathscr{B}^\beta_0)^*$ is isomorphic to $\mathscr{A}^1$ and $(\mathscr{A}^1)^*$ is isomorphic to $\mathscr{B}^\beta$

where $\mathscr{A}^1 = \mathscr{H}(\mathbb{D}) \cap L^1(\mathbb{D}, dA)$ is the classical Bergman space.

Isomorphism

$\mathscr{B}^\beta$ is isomorphic to ℓ^∞ and $\mathscr{B}^\beta_0$ is isomorphic to c_0.

Boundedness on Bloch spaces

The boundedness of any composition operators viewed on (classical) Bloch spaces is clear by the Schwarz-Pick inequality. Indeed

$$\forall z \in \mathbb{D}, \qquad (1-|z|^2)\big|(f\circ\varphi)'(z)\big| = \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2}\cdot(1-|\varphi(z)|^2)\big|f'(\varphi(z))\big|$$

More generally,

Theorem (Contreras-Hernàndez Díaz '00, Xiao '01,...)

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be analytic and let μ, $\beta \in (0,\infty)$. Then

$C_\varphi : \mathscr{B}^\mu \longrightarrow \mathscr{B}^\beta$ is bounded *if and only if* $\displaystyle\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\mu}|\varphi'(z)| < \infty$

$C_\varphi : \mathscr{B}_0^\mu \longrightarrow \mathscr{B}_0^\beta$ is bounded *if and only if* $\varphi \in \mathscr{B}_0^\beta$ and $\displaystyle\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\mu}|\varphi'(z)| < \infty$

Compactness

The characterization of the compactness of composition operators on (classical) Bloch spaces was settled by *Madigan-Matheson* ('95).

More generally,

Theorem (Contreras-Hernàndez Díaz '00, Xiao '01,...)

$C_\varphi : \mathscr{B}^\mu \longrightarrow \mathscr{B}^\beta$ is compact *if and only if* $\varphi \in \mathscr{B}^\beta$ and

$$\lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\mu}|\varphi'(z)| = 0.$$

$C_\varphi : \mathscr{B}^\mu_0 \longrightarrow \mathscr{B}^\beta_0$ is compact *if and only if* $\lim_{|z|\to 1^-} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\mu}|\varphi'(z)| = 0.$

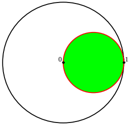
Compactness - examples

1. For $\beta > 0$ and every symbol φ such that $\varphi \in \mathscr{B}^\beta$ and $\|\varphi\|_\infty < 1$, the operator C_φ is compact on $\mathscr{B}^\beta$. (and even nuclear)

¿ Are there examples of **compact** composition operator C_φ with $\overline{\varphi(\mathbb{D})} \cap \mathbb{T} \neq \emptyset$?

Especially in the case of the classical Bloch space $\mathscr{B}$?

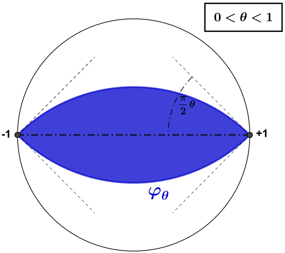
2. Let $\varphi(z) = \dfrac{z+1}{2}$



Then $C_\varphi : \mathscr{B}^\beta \longrightarrow \mathscr{B}^\beta$ is not compact for $\beta > 0$.

The lens map

3. Let $\varphi_\theta(z) = \dfrac{\kappa(z)^\theta - 1}{\kappa(z)^\theta + 1}$ be the lens map , where $\kappa(z) = \dfrac{1+z}{1-z}$.



$$z \in \mathbb{D} \xrightarrow{\kappa} \kappa(z) \xrightarrow{z^\theta} (\kappa(z))^\theta \xrightarrow{\kappa^{-1}} \varphi_\theta(z)$$

(Farès-L.)

- For $0 < \beta < 1$, $C_{\varphi_\theta} : \mathscr{B}^\beta \longrightarrow \mathscr{B}^\beta$ is not bounded (!)
- For $\beta = 1$, $C_{\varphi_\theta} : \mathscr{B} \longrightarrow \mathscr{B}$ is bounded but not compact.
- For $\beta > 1$, $C_{\varphi_\theta} : \mathscr{B}^\beta \longrightarrow \mathscr{B}^\beta$ is compact.

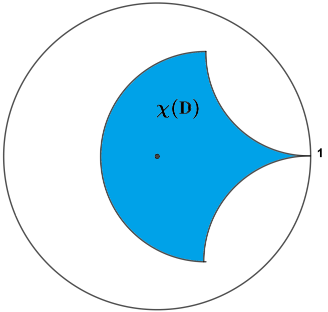
The cusp map

Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ be an analytic univalent map. Assumed that $\overline{\varphi(\mathbb{D})} \cap \mathbb{T} = 1$, the region $\varphi(\mathbb{D})$ is said to have a nontangential cusp at 1 if

$$dist(w, \partial\varphi(\mathbb{D})) = o(|1-w|) \quad as \ w \longrightarrow 1 \ in \ \varphi(\mathbb{D}).$$

$\varphi(\mathbb{D})$ lies inside a Stolz angle if there exist $r, M > 0$ such that

$$|1-w| \leq M(1-|w|^2), \quad if \ |1-w| < r, \ w \in \varphi(\mathbb{D}).$$



Symbols touching the boundary.

(Madigan and Matheson '95)

- If φ is univalent and if $\varphi(\mathbb{D})$ has a nontangential cusp at 1 and touches the unit circle at no other point, then C_φ is a compact operator on $\mathscr{B}_0$.

  Therefore

- The cusp map is a symbol χ such that $\chi(\mathbb{D})$ touches the unit circle and defines a compact composition operator C_χ on $\mathscr{B}$.

A (maybe more) striking example:

(Smith '98)

There exists an inner function (a Blaschke product) φ such that C_φ is a compact composition operator on $\mathscr{B}$.

Nuclear

A natural way to construct a (compact) operator between arbitrary Banach spaces is to consider an ℓ^1-sum of rank 1 operators.

Definition

A linear operator $T : X \longrightarrow Y$ is said to be nuclear when there exist a sequence $(x_n^*) \subset X^*$ and a sequence $(y_n) \subset Y$ such that $\sum_n \|x_n^*\|\|y_n\| < \infty$ and

$$T = \sum_{n=1}^{\infty} x_n^* \otimes y_n,$$

where $x_n^* \otimes y_n(x) = x_n^*(x) y_n$.

Indeed, such operators are compact.

Absolutely summing

Another well known operator ideal is the class of p-summing operators.

Definition

An operator $T : X \longrightarrow Y$ is p-absolutely summing, $1 \leq p < +\infty$ if there exists a constant $C < \infty$ such that for all finite sequences $(x_j)_{j=1}^n \subset X$ we have

$$\Big(\sum_{j=1}^n \|Tx_j\|_Y^p\Big)^{1/p} \leq C \sup_{\|x^*\|\leq 1} \Big(\sum_{j=1}^n |x^*(x_j)|^p\Big)^{1/p} = C \sup_{a\in B_{\ell^{p'}}} \Big\|\sum_{j=1}^n a_j x_j\Big\|_X.$$

Generic example: K a compact space, ν a Borel probability measure on K.

$$j_p : \begin{cases} C(K) & \longrightarrow & L^p(K,\nu) \\ f & \longmapsto & f \end{cases}$$

Indeed, for $(f_j)_{j=1}^N \in C(K)$

$$\Big(\sum_{j=1}^N \|j_p(f_j)\|_{L^p(\nu)}^p\Big)^{1/p} = \Big(\int_K \sum_{j=1}^N |f_j|^p d\nu\Big)^{1/p} \leq \Big(\sup_{w\in K} \sum_{j=1}^N |f_j(w)|^p\Big)^{1/p}$$

Up to restrictions/compositions, it remains p-summing...

Absolutely summing

$T : X \longrightarrow Y$ is 1-summing means that

$$\sum \pm x_n \text{ converges } \implies \sum \|T(x_n)\| < \infty$$

Observe that:

nuclear implies 1-summing

1-summing does not imply compact in general

BUT

When $X = c_0$, then 1-summing implies nuclear (hence compact).

When $X = \ell^\infty$, then 1-summing implies nuclear (hence compact).

Absolutely summing

It is easy to see that

- When $1 \leq p < q < \infty$, $\quad T$ is p-summing $\implies$ T is q-summing.

Pietsch domination theorem

An operator $T : X \longrightarrow Y$ is p-absolutely summing if and only if there exist a constant C and a Borel probability measure ν on $(B_{X^*}, \sigma(X^*, X))$ such that

$$\|T(x)\| \leq C\Big(\int_{B_{X^*}} |\langle x^*, x\rangle|^p d\nu(x^*)\Big)^{1/p}, \quad \forall x \in X.$$

This can be seen as a **factorization** result:

$$i_X(x)(x^*) = x^*(x)$$

$$X \overset{i_X}{\longleftrightarrow} \widetilde{X} \subset C(B_{X^*}) \overset{j_p}{\longrightarrow} \widetilde{X}_p \subset L^p(B_{X^*}, \nu) \overset{\widetilde{T}}{\longrightarrow} Y$$

$$\widetilde{T}(i_X(x)) = T(x).$$

Absolutely summing composition operator on Bloch spaces

We have a complete characterization (Farès-L. '20)

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic, μ, $\beta > 0$ and $p \geq 1$.

The following assertions are equivalent.

(i) The composition operator $C_\varphi : \mathcal{B}^\mu \longrightarrow \mathcal{B}^\beta$ is p-summing.

(ii) The composition operator $C_\varphi : \mathcal{B}_0^\mu \longrightarrow \mathcal{B}^\beta$ is p-summing.

(iii) $$\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \left(\frac{(1-|w|^2)}{|1-\overline{w}\varphi(z)|}\right)^{2p} \left(\frac{(1-|z|^2)^\beta|\varphi'(z)|}{|1-\overline{w}\varphi(z)|^\mu}\right)^p \frac{dA(w)}{(1-|w|^2)^2} < +\infty.$$

Moreover, when $\varphi \in \mathcal{B}_0^\beta$, the preceding assertions are also equivalent to

(iv) The composition operator $C_\varphi : \mathcal{B}_0^\mu \longrightarrow \mathcal{B}_0^\beta$ is p-summing.

The case $p = 1$ gives a characterization of nuclear composition operators on Bloch spaces and extends *Farès-L '19* ($p = \mu = \beta = 1$).

Absolutely summing

Sketch of proof: *how to get the condition...*

Use Pietsch theorem: for some probability measure ν on $\left(B_{(\mathscr{B}_0^\mu)^*}, \sigma((\mathscr{B}_0^\mu)^*, \mathscr{B}_0^\mu)\right)$,

$$\|C_\varphi(f)\|^p_{\mathscr{B}^\beta} \leq \pi_p^p(C_\varphi) \int_{B_{(\mathscr{B}_0^\mu)^*}} |\xi(f)|^p \, d\nu(\xi) \qquad \text{for every } f \in \mathscr{B}_0^\mu$$

There exists $\alpha \geq 1$ satisfying: for every $\xi \in B_{(\mathscr{B}_0^\mu)^*}$, there exists $h \in \alpha B_{\mathscr{A}^1}$ such that

$$\xi(f) = \langle h, f \rangle \qquad \text{for any } f \in \mathscr{B}_0^\mu.$$

Apply this with $f_w(z) = \dfrac{(1-|w|^2)^{2/p'}}{(1-\overline{w}z)^{1+\mu}} \in \mathscr{B}_0^\mu \cap H^\infty$:

$$\left|\xi(f_w)\right|^p = (1-|w|^2)^{2(p-1)}|h(w)|^p \leq |h(w)|.\|h\|^{(p-1)}_{\mathscr{A}^1}.$$

Then integrating over $\mathbb{D}$

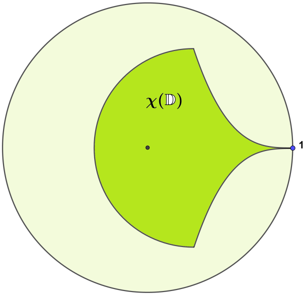
$$\int_{\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{|w|^p(1-|w|^2)^{2(p-1)}(1-|z|^2)^{\beta p}|\varphi'(z)|^p}{|1-\overline{w}\varphi(z)|^{(2+\mu)p}} dA(w) \lesssim \pi_p^p(C_\varphi) \sup_{h\in\alpha B_{\mathscr{A}^1}} \|h\|^p_{\mathscr{A}^1}.$$

Some other examples

Using our characterization, we can produce some examples with particular behavior:

There exists a symbol φ such that $\varphi(\mathbb{D})$ touches the unit circle and C_φ defines a nuclear operator on $\mathscr{B}$.

Taking a cusp map φ more flattened than the one of Madigan and Matheson, satisfying: $dist(z, \partial\varphi(\mathbb{D})) = O(|1-z|^3)$, for every $z \in \varphi(\mathbb{D})$



Some other examples

Another striking example:

Let $\beta \geq 1$.

- There exists a symbol ϕ which is inner (a Blaschke product) such that C_ϕ is nuclear on $\mathscr{B}^\beta$.
- This is impossible when $\beta < 1$.

This a direct consequence from our characterization and a result due to Aleksandrov-Anderson-Nicolau ('99):

there are some Blaschke product ϕ satisfying

$$\forall z \in \mathbb{D}, \qquad (1-|z|^2)|\phi'(z)| \leq \left(1-|\phi(z)|^2\right)^3.$$

Some other examples

There exists a compact composition operator on $\mathscr{B}$ which is not p-summing for any $p \geq 1$.

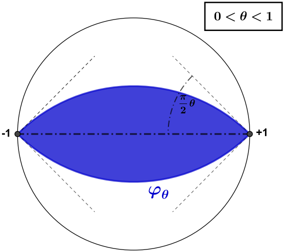
We consider a cusp map Φ such that its domain $\Phi(\mathbb{D})$ is bounded by some convex curves of type $\gamma_1(t) = (1 - t, \frac{t}{\theta(t)})$ and $\gamma_2(t) = (1 - t, -\frac{t}{\theta(t)})$, for t in a neighborhood of 0 and $\theta : (0,1) \longrightarrow (0, +\infty)$, such that $\theta(t)$ tends to infinity when t tends to 0:
For any $p \geq 1$,

$$\int_0^{1/2} \frac{1}{s\theta(s)^{p+1}} ds = \infty,$$

For a concrete example, just choose $\theta(t) = \ln(\ln(\frac{1}{t}))$, for $t < e^{-1}$.

Lens map semi-group

Recall



$z \in \mathbb{D} \xrightarrow{\kappa} \kappa(z) \xrightarrow{Z^\theta} (\kappa(z))^\theta \xrightarrow{\kappa^{-1}} \varphi_\theta(z)$

Therefore

$$\varphi_{\theta'} \circ \varphi_\theta = \varphi_{\theta\theta'}$$

So,

we can embed a composition operator C_{φ_θ} in a semi-group $L_t = C_{\varphi_{\theta^t}}$ (where $t > 0$).

Lens map semi-group

For $p \geq 1$ and $\beta > 1$

The lens map φ_θ induces a p-summing composition operator on $\mathscr{B}^\beta$ for p large enough:

$$\text{it is sufficient that} \quad p > \frac{\theta}{(\beta-1)(1-\theta)},$$

In particular, if $\theta < 1 - \frac{1}{\beta}$, then C_{φ_θ} is nuclear on $\mathscr{B}^\beta$.

Therefore

The lens map semi-group is eventually p-summing. More precisely,

- L_t is p-summing for $t > \dfrac{\ln\left(1 - \frac{1}{1+p(\beta-1)}\right)}{\ln(\theta)}$
- L_t is nuclear for $t > \dfrac{\ln\left(1 - \frac{1}{\beta}\right)}{\ln(\theta)}$

Merci !