A family of Möbius invariant function spaces

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The talk consists of the following two parts:

- Boundary multipliers of a family of Möbius invariant function spaces $F(p, p-2, s)$, which is based on a joint work with J. Pau.
- Intersections and unions of a family of Möbius invariant function spaces $F(p, p-2, s)$, which is based on a joint work with H. Wulan and F. Ye.

Notation

- $\mathbb{D} = \{z : |z| < 1\}$, the open unit disk in the complex plane $\mathbb{C}$.
- $\mathbb{T} = \{z : |z| = 1\}$, the unit circle.
- $dA(z) = dxdy$, $z = x + iy$.
- $H(\mathbb{D})$ is the space of all analytic functions on $\mathbb{D}$.
- $\mathrm{Aut}(\mathbb{D})$: the Möbius group, that is the set of all conformal automorphisms of $\mathbb{D}$. Each element of $\mathrm{Aut}(\mathbb{D})$ is a fractional transformation ϕ of the following form

$$\phi(z) = e^{i\theta}\sigma_a(z), \quad \sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad a \in \mathbb{D}.$$

A general family of function spaces.

For $0 < p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$, the space $F(p,q,s)$ is the set of those functions $f \in H(\mathbb{D})$ with

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) < \infty.$$

- R. Zhao, On a general family of function spaces, Ann. Acad. Sci. Fenn. Math. Diss., 105 (1996), 56 pp.
- $F(p,q,s)$ contains only constant functions if $s+q \leq -1$.
- $A^p_q = F(p, p+q, 0)$ when $p > 0$ and $q > -1$.
- $F(p, p-2, s)$ is Möbius invariant in the sense that

$$\|f\circ\phi\|_{F(p,p-2,s)} = \|f\|_{F(p,p-2,s)}$$

for every $f \in F(p,p-2,s)$ and $\phi \in \mathrm{Aut}(\mathbb{D})$.

The Möbius invariant space $F(p, p-2, s)$, $p > 0$ and $s > 0$.

- $F(p, p-2, s)$ contains only constant functions if $s + p \leq 1$.
- If $s > 1$, all $F(p, p-2, s)$ spaces are the same and equal to the Bloch space $\mathcal{B}$.

$F(2, 0, s)$ is the Möbius invariant space $\mathcal{Q}_s$.

- R. Aulaskari, J. Xiao and R. Zhao, On subspaces and subsets of *BMOA* and *UBC*, *Analysis*, **15** (1995), 101-121.
- J. Xiao, Holomorphic $\mathcal{Q}$ Classes, Springer, LNM 1767, Berlin, 2001.
- J. Xiao, Geometric $\mathcal{Q}_p$ functions, Birkhäuser Verlag, Basel-Boston-Berlin, 2006.

An important application of $F(p,q,s)$ spaces

- A. Wynn, Counterexamples to the discrete and continuous weighted Weiss conjectures, SIAM J. Control Optim., 48 (2009), 2620-2635.

In 2009, A. Wynn gave nice connection between $F(p,q,s)$ spaces and weighed Weiss conjecture in the field of linear system theory and cybernetics.

$\mathcal{Q}_s$ and $\mathcal{Q}_s(\mathbb{T})$ spaces

- For $0 < s < \infty$, $\mathcal{Q}_s(\mathbb{T})$ is the space of functions $f \in L^2(\mathbb{T})$ with

$$\|f\|^2_{\mathcal{Q}_s(\mathbb{T})} = \sup_{I \subseteq \mathbb{T}} \frac{1}{|I|^s} \int_I \int_I \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{2-s}} |d\zeta||d\eta| < \infty,$$

where $|I|$ is the length of an arc I of the unit circle $\mathbb{T}$.

- $\mathcal{Q}_s(\mathbb{T}) = BMO(\mathbb{T})$ if and only if $s > 1$.

Theorem (M. Essén, J. Xiao, J. Reine Angew. Math., 1997)

If $0 < s < 1$ and f is in the Hardy space H^1, then $f \in \mathcal{Q}_s$ if and only if $f^* \in \mathcal{Q}_s(\mathbb{T})$. Here f^* is the non-tangential limit of f.

For Banach function spaces X and Y, denote by $M(X, Y)$ the class of all pointwise multipliers from X to Y. Namely,

$$M(X, Y) = \{f : fg \in Y \text{ for all } g \in X\}.$$

If $X = Y$, we just write $M(X, Y)$ as $M(X)$ for the collection of multipliers of X.

Multipliers of $BMO(\mathbb{T})$ and $BMOA$

- D. Stegenga, Amer. J. Math., 1976
  $f \in M(BMO(\mathbb{T}))$ *if and only if* $f \in L^\infty$ *and*
  $$\sup_{I \subseteq \mathbb{T}} \left(\log \frac{2}{|I|}\right)^2 \frac{1}{|I|^2} \int_I \int_I |f(\zeta) - f(\eta)|^2 |d\zeta||d\eta| < \infty.$$
- J. Ortega, J. Fàbrega, Ann. Inst. Fourier , 1996
  $f \in M(BMOA)$ *if and only if* $f \in H^\infty$ *and*
  $$\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|}\right)^2 \int_{\mathbb{D}} |f'(z)|^2 \left(1 - |\sigma_a(z)|^2\right) dA(z) < \infty.$$

Conjecture (J. Xiao, Pacific J. Math., 2000)

Let $0 < s < 1$. *Then* $f \in M(\mathcal{Q}_s)$ *if and only if* $f \in H^\infty$ *and*

$$\sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|}\right)^2 \int_{\mathbb{D}} |f'(z)|^2 \left(1-|\sigma_a(z)|^2\right)^s dA(z) < \infty.$$

This conjecture was proved by J. Pau and J.A. Peláez in 2009:
J. Pau and J.A. Peláez, Multipliers of Möbius invariant $\mathcal{Q}_s$ spaces, *Math. Z.* **261** (2009), 545–555.
Furthermore, the characterization of multipliers of $F(p, p-2, s)$ spaces was known in the following paper:
J. Pau and R. Zhao, Carleson measures, Riemann-Stieltjes and multiplication operators on a general family of function spaces, *Integral Equations Operator Theory* **78** (2014), 483–514.

J. Xiao's Conjecture on $M(\mathcal{Q}_s(\mathbb{T}))$

Conjecture (J. Xiao, Pacific J. Math., 2000)

Let $0 < s < 1$. *Then* $f \in M(\mathcal{Q}_s(\mathbb{T}))$ *if and only if* $f \in L^\infty$ *and*

$$\sup_{I \subseteq \mathbb{T}} \frac{1}{|I|^s} \left(\log \frac{2}{|I|} \right)^2 \int_I \int_I \frac{|f(\zeta) - f(\eta)|^2}{|\zeta - \eta|^{2-s}} |d\zeta||d\eta| < \infty.$$

For $1 < p < \infty$ and $s > 0$, the spaces $\mathcal{Q}_s^p(\mathbb{T})$ consisting of functions $f \in L^p(\mathbb{T})$ such that

$$\|f\|_{\mathcal{Q}_s^p(\mathbb{T})}^p = \sup_{I \subseteq \mathbb{T}} \frac{1}{|I|^s} \int_I \int_I \frac{|f(\zeta) - f(\eta)|^p}{|\zeta - \eta|^{2-s}} |d\zeta||d\eta| < \infty.$$

- If $p = 2$, then $\mathcal{Q}_s^p(\mathbb{T}) = \mathcal{Q}_s(\mathbb{T})$.

Theorem (G. Bao, J. Pau, Ann. Acad. Sci. Fenn. Math., 2016)

Let $p > 1$ and $0 < s < 1$. Suppose $f \in H^p$. Then $f^* \in \mathcal{Q}_s^p(\mathbb{T})$ if and only if $f \in F(p, p-2, s)$.

Theorem (G. Bao, J. Pau, 2016)

Let $p > 1$ and $0 < s < 1$. Suppose $f \in L^p(\mathbb{T})$. The following conditions are equivalent.

(a) $f \in \mathcal{Q}_s^p(\mathbb{T})$.

(b) $|\nabla \widehat{f}(z)|^p (1-|z|^2)^{p-2+s} dA(z)$ *is an s-Carleson measure.*

(c)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(\zeta)-f(\eta)|^p}{|\zeta-\eta|^{2-s}} \left(\frac{1-|a|^2}{|\zeta-a||\eta-a|}\right)^s |d\zeta||d\eta| < \infty.$$

For $s > 0$, a positive Borel measure μ on $\mathbb{D}$ is called an s-Carleson measure if $\mu(s(I)) \leq C|I|^s$ for any $I \subseteq \mathbb{T}$, where $S(I)$ is the usual Carleson box based on I.

Inclusion relations

Theorem (G. Bao, J. Pau, 2016)

Let $1 < p_1, p_2 < \infty$ *and* $0 < s, r < 1$.

(1) *If* $p_1 \leq p_2$, *then* $\mathcal{Q}_s^{p_1}(\mathbb{T}) \subseteq \mathcal{Q}_r^{p_2}(\mathbb{T})$ *if and only if* $s \leq r$.

(2) *If* $p_1 > p_2$, *then* $\mathcal{Q}_s^{p_1}(\mathbb{T}) \subseteq \mathcal{Q}_r^{p_2}(\mathbb{T})$ *if and only if* $\frac{1-s}{p_1} > \frac{1-r}{p_2}$.

Theorem (G. Bao, J. Pau, 2016)

Let $1 < p_1, p_2 < \infty$ *and* $0 < s, r < 1$. *Then the following are true.*

(1) *If* $p_1 \le p_2$ *and* $s \le r$, *then* $f \in M(\mathcal{Q}_s^{p_1}(\mathbb{T}), \mathcal{Q}_r^{p_2}(\mathbb{T}))$ *if and only if* $f \in L^\infty(\mathbb{T})$ *and*

$$\sup_{I \subseteq \mathbb{T}} \frac{1}{|I|^r} \left(\log \frac{2}{|I|}\right)^{p_2} \int_I \int_I \frac{|f(\zeta) - f(\eta)|^{p_2}}{|\zeta - \eta|^{2-r}} |d\zeta||d\eta| < \infty. \qquad (A)$$

(2) *Let* $p_1 > p_2$ *and* $s \le r$. *If* $\frac{1-s}{p_1} > \frac{1-r}{p_2}$, *then* $f \in M(\mathcal{Q}_s^{p_1}(\mathbb{T}), \mathcal{Q}_r^{p_2}(\mathbb{T}))$ *if and only if* $f \in L^\infty(\mathbb{T})$ *and* f *satisfies (A). If* $\frac{1-s}{p_1} \le \frac{1-r}{p_2}$, *then* $M(\mathcal{Q}_s^{p_1}(\mathbb{T}), \mathcal{Q}_r^{p_2}(\mathbb{T})) = \{0\}$.

(3) *If* $s > r$, *then* $M(\mathcal{Q}_s^{p_1}(\mathbb{T}), \mathcal{Q}_r^{p_2}(\mathbb{T})) = \{0\}$.

An observation. Suppose g is real valued. Let $\widetilde{g}$ be the harmonic conjugate function of $\widehat{g}$. Set $h = \widehat{g} + i\widetilde{g}$. The Cauchy-Riemann equations give $|\nabla \widehat{g}(z)| \approx |h'(z)|$.

Lemma (G. Bao, J. Pau, 2016)

Let $1 < p_1,\ p_2 < \infty$ *and* $0 < s,\ r < 1$. *If there exists a lacunary Fourier series* $g(e^{i\theta}) = \sum_{k=0}^{\infty} a_k e^{i2^k\theta} \in \mathcal{Q}_s^{p_1}(\mathbb{T}) \setminus \mathcal{Q}_r^{p_2}(\mathbb{T})$, *then* $M(\mathcal{Q}_s^{p_1}(\mathbb{T}),\ \mathcal{Q}_r^{p_2}(\mathbb{T})) = \{0\}$.

However this lemma miss the case that $p_1 < p_2$, $s > r$ and $\frac{1-s}{p_1} \geq \frac{1-r}{p_2}$. In this case, $\mathcal{Q}_s^{p_1}(\mathbb{T}) \not\subseteq \mathcal{Q}_r^{p_2}(\mathbb{T})$, but any lacunary Fourier series in $\mathcal{Q}_s^{p_1}(\mathbb{T})$ must be in $\mathcal{Q}_r^{p_2}(\mathbb{T})$. Thus, we are in need to look for another method to determine that $M(\mathcal{Q}_s^{p_1}(\mathbb{T}), \mathcal{Q}_r^{p_2}(\mathbb{T}))$ is trivial in this case.

- To handle part (3), for every $\zeta \in \mathbb{T}$, we need to construct a Blaschke sequence $\{a_k\}$ converging to ζ in a way that

$$B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \overline{a_k} z} \in F(p_1, p_1 - 2, s) \setminus F(p_2, p_2 - 2, r).$$

- By a result in
  D. Girela, J.A. Peláez and D. Vukotić, *Complex Var. Elliptic Equ.* **52** (2007), 161–173,
  this is not possible if the sequence $\{a_k\}$ converges to ζ non-tangentiall

For $c > 0$ and $\delta > 1$, consider the region

$$\Omega_{\delta,c}(\theta) = \left\{ re^{i\varphi} \in \mathbb{D} : 1 - r > c \left| \sin \frac{\varphi - \theta}{2} \right|^{\delta} \right\}.$$

Then $\Omega_{\delta,c}(\theta)$ touches $\mathbb{T}$ at $e^{i\theta}$ tangentially. We say that a function h, defined in $\mathbb{D}$, has Ω_δ-limit L at $e^{i\theta}$ if $h(z) \to L$ as $z \to e^{i\theta}$ within $\Omega_{\delta,c}(\theta)$ for every c.

A. Nagel, W. Rudin and J. Shapiro, Tangential boundary behavior of function in Dirichlet-type spaces, *Ann. of Math.* **116** (1982), 331–360.

Theorem (A. Nagel, W. Rudin, J. Shapiro, Ann. of Math., 1982)

Suppose $1 \le p < \infty$, $f \in L^p(\mathbb{T})$, $0 < \alpha < 1$, *and*

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})d\theta}{(1 - e^{-i\theta}z)^{1-\alpha}}, \quad z \in \mathbb{D}.$$

If $\alpha p < 1$ *and* $\delta = 1/(1 - \alpha p)$, *then the* Ω_δ*-limit of* h *exists almost everywhere on* $\mathbb{T}$.

Proposition

Let $1 < p < \infty$ and $0 < s < t < 1$. Suppose $h \in B_p(s)$. Then the $\Omega_{1/t}$-limit of h exists almost everywhere on $\mathbb{T}$.

Here $B_p(s)$ is the space of those functions $f \in H(\mathbb{D})$ with

$$\|f\|_{B_p(s)} = \left(\int_{\mathbb{D}} |f'(z)|^p \, (1 - |z|^2)^{p-2+s} \, dA(z) \right)^{1/p} < \infty,$$

Theorem (F. Pérez-González and J. Rättyä, Proc. Edinb. Math. Soc., 2009)

Let $0 < s < 1$ *and* $p > \max\{s, 1-s\}$. *Then an inner function belongs to* $F(p, p-2, s)$ *if and only if it is a Blaschke product associated with a sequence* $\{z_k\}_{k=1}^{\infty} \subseteq \mathbb{D}$ *which satisfies that* $\sum_k (1-|z_k|)^s \delta_{z_k}$ *is an s-Carleson measure; that is,*

$$\sup_{a\in\mathbb{D}} \sum_{k=1}^{\infty} \left(1-|\sigma_a(z_k)|^2\right)^s < \infty.$$

The following lemma is key to prove part (3) of our main Theorem.

Lemma (G. Bao, J. Pau, 2016)

Let $0 < r < s < 1$ and $0 < t < 1$. For every $e^{i\theta} \in \mathbb{T}$, there exists a Blaschke sequence $\{a_k\}_{k=1}^{\infty}$ satisfying the following conditions.

(a) $e^{i\theta}$ *is the unique accumulation point of* $\{a_k\}$.

(b) $\{a_k\} \subseteq \Omega_{1/t,c}(\theta)$ *for some* $c > 0$.

(c) $\sum_k (1 - |a_k|)^s \delta_{a_k}$ *is an s-Carleson measure.*

(d) $\sum_k (1 - |a_k|)^r \delta_{a_k}$ *is not an r-Carleson measure.*

It remans open to characterize multipliers of $F(p,q,s)$ spaces for certain range of parameters p, q and s. We refer to the following paper:

J. Pau and R. Zhao, Carleson measures, Riemann-Stieltjes and multiplication operators on a general family of function spaces, *Integral Equations Operator Theory* **78** (2014), 483–514.

Part II: Intersections and unions of a family of Möbius invariant function spaces $F(p, p-2, s)$, which is based on a joint work with H. Wulan and F. Ye.

Intersections and unions of certain analytic function spaces usually appears in the study of complex linear differential equations in the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$.

The order of growth of analytic functions

The order of growth of $f \in H(\mathbb{D})$ can be defined as

$$\sigma(f) =: \limsup_{r \to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)},$$

where $T(r,f)$ is the the Nevanlinna characteristic.

Consider the following complex differential equation:

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + a_0(z)f = 0, \qquad (E_1)$$

where k is a positive integer and the coefficients $a_j \in H(\mathbb{D})$ for all j.

Theorem (R. Korhonen and J. Rättyä, JMAA, 2009)

Let $0 \le \alpha < \infty$. Then all solutions f of (E_1) satisfies $\sigma(f) \le \alpha$ if and only if $a_j \in \bigcap_{0<p<\frac{1}{k-j}} A^p_\alpha$ for all $j = 0, \cdots, k-1$.

Second order complex differential equation

We consider

$$f'' + Af = 0. \qquad (E_2)$$

It is known that if $A \in H(\mathbb{D})$, then all solutions of (E_2) belong to $H(\mathbb{D})$ as well.

Solutions of (E_2) having pre-given zeros

Given a sequence $\{a_k\}$ of distinct points on $\mathbb{D}$, one can ask whether there exists a function $A \in H(\mathbb{D})$ such that (2) has a solution f with zeros precisely at the points a_k. In this case, $\{a_k\}$ is said to be a pre-given zero sequence.

- J. Heittokangas, Solutions of $f'' + A(z)f = 0$ in the unit disc having Blaschke sequences as the zeros, Comput. Methods Funct. Theory, 5 (2005), 49-63.
- J. Heittokangas, A survey on Blaschke-oscillatory differential equations, with updates, Blaschke products and their applications, 43-98, Fields Inst. Commun., 65, Springer, New York, 2013.

Separated sequence

A sequence $\{a_k\}$ in $\mathbb{D}$ is said to be separated if

$$\inf_{n\neq k}\left|\frac{a_n-a_k}{1-\overline{a_n}a_k}\right|>0.$$

Let δ_z be the unit point-mass measure at $z \in \mathbb{D}$.

Theorem (J. Gröhn, Constr. Approx., 2019)

Let $0 < s \leq 1$. If $\Lambda \subset \mathbb{D}$ is a separated sequence such that $\sum_{z_n \in \Lambda}(1-|z_n|)^s \delta_{z_n}$ is an s-Carleson measure, then there exists a function A analytic in $\mathbb{D}$ such that $|A(z)|^2(1-|z|^2)^{2+s}dm(z)$ is an s-Carleson measure and the equation $f'' + Af = 0$ admits a nontrivial solution $f \in \mathcal{Q}_s \cap H^\infty$ whose zero-sequence is Λ.

A further result

Theorem (F. Ye, Comput. Methods Funct. Theory, 2019)

Let $0 < s < 1$ and $1 < q < \infty$. If $\Lambda \subset \mathbb{D}$ is a separated sequence such that $\sum_{z_n \in \Lambda}(1 - |z_n|)^s \delta_{z_n}$ is an s-Carleson measure, then there exists a function A analytic in $\mathbb{D}$ such that $|A(z)|^q(1 - |z|^2)^{2q-2+s}dm(z)$ is an s-Carleson measure and the equation $f'' + Af = 0$ admits a nontrivial solution $f \in \bigcap_{1<p<\infty}(F(p, p-2, s) \cap H^\infty)$ whose zero-sequence is Λ.

For $0 < s < 1$ and $2 < q < \infty$, if $|A(z)|^2(1 - |z|^2)^{2+s}dm(z)$is an s-Carleson measure, then $|A(z)|^q(1 - |z|^2)^{2q-2+s}dm(z)$ is an s-Carleson measure.

When $0 < s < 1 < p < 2$, $F(p, p-2, s) \cap H^\infty \subsetneq \mathcal{Q}_s \cap H^\infty$.

Theorem (R. Korhonen and J. Rättyä, CMFT, 2005)

Let $0 < p < \infty$ *and* $-1 < q < \infty$. *Then*

$$\bigcap_{0<p^*<p} \bigcap_{q<q^*<\infty} A^{p^*}_{q^*} = \bigcap_{q<q^*<\infty} A^{p}_{q^*} = \bigcap_{0<p^*<p} A^{p^*}_{q}$$

and

$$\bigcup_{p<p^*<\infty} \bigcup_{-1<q^*<q} A^{p^*}_{q^*} = \bigcup_{-1<q^*<q} A^{p}_{q^*} = \bigcup_{p<p^*<\infty} A^{p^*}_{q}.$$

Intersections and unions of $F(p,q,s)$ spaces

Theorem (R. Korhonen and J. Rättyä, CMFT, 2005)

Let $0 < p < \infty$, $-2 < q < \infty$ *and* $0 \leq s < \infty$ *such that* $q + s > -1$. *Then*

$$\bigcap_{0<p^*<p} \bigcap_{q<q^*<\infty} F(p^*, q^*, s) = \bigcap_{q<q^*<\infty} F(p, q^*, s) = \bigcap_{0<p^*<p} F(p*, q, s)$$

and

$$\bigcup_{p<p^*<\infty} \bigcup_{-2<q^*<q} F(p^*, q^*, s) = \bigcup_{-2<q^*<q} F(p, q^*, s) = \bigcup_{p<p^*<\infty} F(p^*, q, s).$$

Intersections and unions of $F(p,q,s)$ spaces

Theorem (R. Korhonen and J. Rättyä, CMFT, 2005)

Let $0 < p < \infty$, $-2 < q < \infty$ *and* $0 \le s < \infty$ *such that* $q + s > -1$. *Then*

$$\bigcap_{0<p^*<p} F(p^*, p^* + q, s) = \bigcap_{0<p^*<p} F(p*, p + q, s)$$

and

$$\bigcup_{p<p^*<\infty} F(p^*, p^* + q, s) = \bigcup_{p<p^*<\infty} F(p^*, p + q, s).$$

Theorem (R. Korhonen and J. Rättyä, CMFT, 2005)

Let $0 < p^* < \infty$, $0 < \alpha < \infty$ *and* $0 \le s^* \le 1$ *with* $\alpha p^* + s^* > 1$. *Then*

$$\bigcap_{p^* < p < \infty} F(p, \alpha p - 2, s^*) \subseteq \bigcap_{s^* < s < \infty} F(p^*, \alpha p^* - 2, s) \tag{1}$$

and

$$\bigcup_{0 \le s < s^*} F(p^*, \alpha p^* - 2, s) \subseteq \bigcup_{0 < p < p^*} F(p, \alpha p - 2, s^*). \tag{2}$$

R. Korhonen and J. Rättyä proved that if $s^* = 0$ or $s^* = 1$, then the inclusion (1) is strict. If $s^* = 1$, then the inclusion (2) is also strict. They asked that whether the inclusions (1) and (2) are strict when $s^* \in (0, 1)$.

K-Carleson measures

- M. Essén, H. Wulan and J. Xiao, Several function-theoretic characterizations of Möbius invariant $\mathcal{Q}_K$ spaces, *J. Funct. Anal.*, **230** (2006), 78-115.

Let $K : (0,\infty) \to (0,\infty)$ be a nondecreasing function. Suppose μ is a nonnegative Borel measure on $\mathbb{D}$. Then μ is called a K-Carleson measure if

$$\sup_{I\subseteq\mathbb{T}} \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) < \infty.$$

If $K(t) = t^s$, $0 < s < \infty$, then μ is a K-Carleson measure if and only if the measure $(1-|z|^2)^s d\mu(z)$ is an s-Carleson measure.
K-Carleson measures depends only on the properties of the function K near zero. For the convenience, we can assume that $K(t) = K(1)$ when $t \geq 1$.

Set

$$\varphi_K(r) = \sup_{0<t\le 1} \frac{K(rt)}{K(t)}, \quad 0 < r < \infty.$$

We will need the following constraint on K:

$$\int_0^1 \varphi_K(r)\frac{dr}{r} < \infty. \tag{3}$$

We say that K satisfies the doubling condition if there exist positive constants C and M such that

$$K(t) \le K(2t) \le CK(t), \quad 0 < t \le M.$$

Theorem (M. Essén, H. Wulan and J. Xiao, JFA, 2006)

Suppose K is a nondecreasing positive function on $(0,\infty)$ satisfying condition (3) and the doubling condition. Let

$$\mu = \sum_{n=1}^{\infty} c_n \delta_{z_n},$$

where $\{c_n\}$ is a sequence of positive numbers. Then the following conditions are equivalent.

(1) *μ is a K-Carleson measure.*

(2)

$$\sup_{a\in\mathbb{D}} \sum_{n=1}^{\infty} c_n K\left(1-|\sigma_a(z_n)|^2\right) < \infty.$$

Theorem (F. Pérez-González and J. Rättyä, Proc. Edinb. Math. Soc., 2009)

Let $0 < s < 1$ *and* $p > \max\{s, 1-s\}$. *Then an inner function belongs to* $F(p, p-2, s)$ *if and only if it is a Blaschke product associated with a sequence* $\{z_k\}_{k=1}^{\infty} \subseteq \mathbb{D}$ *which satisfies that* $\sum_k (1-|z_k|)^s \delta_{z_k}$ *is an s-Carleson measure; that is,*

$$\sup_{a \in \mathbb{D}} \sum_{k=1}^{\infty} \left(1 - |\sigma_a(z_k)|^2\right)^s < \infty.$$

Lemma (G. Bao, H. Wulan, F. Ye, PAMS, 2021)

Let $0 < s < \infty$, $1 < p < \infty$ *and* $e^{i\theta} \in \mathbb{T}$. *Suppose*

$$K(t) = \begin{cases} t^s \left(\ln \frac{2}{t}\right)^{-p}, & t \in (0,1) \\ (\ln 2)^{-p}, & t \geq 1. \end{cases}$$

Then there exists a sequence $\{z_k\}_{k=1}^{\infty}$ *in* $\mathbb{D}$ *satisfying the following conditions.*

(1) $e^{i\theta}$ *is the unique accumulation point of* $\{z_k\}_{k=1}^{\infty}$.

(2) $\sum_{k=1}^{\infty}(1-|z_k|)^s \delta_{z_k}$ *is not an* s*-Carleson measure.*

(3) $\sum_{k=1}^{\infty} \delta_{z_k}$ *is a* K*-Carleson measure.*

Set

$$f(x) = x^{-s} e^{(4\pi x)^{-\alpha}}, \;\; x > 0,$$

where $\alpha \geq \frac{s}{p-1}$. Then f is a strictly decreasing function on $(0, +\infty)$. As usual, denote by f^{-1} the inverse function of f. Define

$$z_k = \left(1 - k^{-\frac{1}{s}}\right) e^{i\theta_k}, \;\; k = 1, 2, 3, \cdots,$$

where

$$\theta_k = \theta + f^{-1}(k).$$

Lemma (G. Bao, H. Wulan, F. Ye, 2021)

Let $0 < s \leq 1$, $0 < p < \infty$ *and* $e^{i\theta} \in \mathbb{T}$. *Suppose*

$$K(t) = \begin{cases} t^s \left(\ln \frac{e^{p/s}}{t}\right)^p, & t \in (0,1), \\ (\frac{p}{s})^p, & t \geq 1. \end{cases}$$

Then there exists a sequence $\{a_k\}_{k=1}^{\infty}$ *in* $\mathbb{D}$ *satisfying the following conditions.*

(1) $e^{i\theta}$ *is the unique accumulation point of* $\{a_k\}_{k=1}^{\infty}$.

(2) $\sum_{k=1}^{\infty}(1-|a_k|)^s \delta_{a_k}$ *is an* s*-Carleson measure.*

(3) $\sum_{k=1}^{\infty} \delta_{a_k}$ *is not a* K*-Carleson measure.*

Set

$$a_k = \left[1 - k^{-\frac{1}{s}} \left(\ln(k+2)\right)^{-\alpha}\right] e^{i\theta_k}, \quad k = 1, 2, 3, \cdots ,$$

where $\frac{1}{s} < \alpha \leq \frac{p+1}{s}$ and

$$\theta_k = \theta + \sum_{j=k}^{\infty} j^{-1} [\ln(j+2)]^{-\alpha s}.$$

We answer partially R. Korhonen and J. Rättyä's question as follows.

Theorem (G. Bao, H. Wulan, F. Ye, 2021)

Let $0 < s^* < 1$ *and* $p^* > \max\{s^*, 1 - s^*\}$. *Then*

$$\bigcap_{p^* < p < \infty} F(p, p-2, s^*) \subsetneqq \bigcap_{s^* < s < \infty} F(p^*, p^*-2, s)$$

and

$$\bigcup_{0 \le s < s^*} F(p^*, p^*-2, s) \subsetneqq \bigcup_{0 < p < p^*} F(p, p-2, s^*).$$

Many thanks for your attention!