On the spectrum of the Hilbert matrix

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Hankel operators on $\ell^2$

Space of square-summable (complex) sequences:

$$\ell^2 = \{(x_n)_n : \|(x_n)\|_2 = \left(\sum_n |x_n|^2\right)^{1/2} < \infty\}.$$ 

**Definition.** A *Hankel operator* on the space $\ell^2$ is an operator defined by a matrix whose entries $a_{m,n}$ depend only on the sum of the coordinates: $a_{m,n} = c_{m+n}$ for some sequence $(c_k)_{k=0}^{\infty}$:

$$
\begin{bmatrix}
c_0 & c_1 & c_2 & \ldots \\
c_1 & c_2 & c_3 & \ldots \\
c_2 & c_3 & c_4 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$
Nehari’s $\ell^2$-theorem

Fourier coefficients of $g \in L^\infty(0, 2\pi)$:

$$\hat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} g(t) dt , \quad n \in \mathbb{Z}.$$  

**Theorem** (Nehari, 1957). Let $H_a$ be a Hankel operator defined by the matrix with entries $a_{m,n} = c_{m+n}$. Then $H_a$ is bounded on $\ell^2$ if and only if there exists a function $g \in L^\infty(0, 2\pi)$ such that $c_k = \hat{g}(k)$, $k \geq 0$.

$$\|H_a\|_{\ell^2 \to \ell^2} = \inf \|g\|_{L^\infty(T)} ,$$

where the infimum is taken over all functions $g$ as above.
The Hilbert matrix $H$

Infinite matrix with entries $a_{m,n} = (m + n + 1)^{-1}$, $m, n \geq 0$:

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Action on an infinite sequence $x = (x_n)_{n=0}^{\infty}$: the $n$-th coordinate is

$$(Hx)_n = \left( \sum_{n=0}^{\infty} \frac{x_n}{m + n + 1} \right)_{m=0}^{\infty}.$$
Basic facts about the Hilbert matrix

- It was introduced by Hilbert in 1894 in the study of orthogonal polynomials.
- It arises naturally as the Gram matrix of the system \((x^n)_{n=0}^{\infty}\) in \(L^2[0, 1]\): \(\langle x^m, x^n \rangle = (m + n + 1)^{-1}\).
- It is a prototype of a Hankel operator on \(\ell^2\): 
  \[a_{m,n} = (m + n + 1)^{-1}\].
- It is a bounded and self-adjoint operator on \(\ell^2\).
- Its norm on \(\ell^2\) can be computed using Nehari’s theorem or Hilbert’s inequality.
Norm of $H$ on $\ell^2$

For the Hilbert matrix $H$, the choice

$$g(t) = ie^{-it}(\pi - t), \quad 0 \leq t < 2\pi,$$

in Nehari’s theorem yields

$$\|H\|_{\ell^2 \to \ell^2} \leq \|g\|_{L^\infty(\mathbb{T})} = \pi.$$

It can be shown that the norm is exactly $\pi$.

Although the general Nehari theorem was discovered much later, the value of $\|H\|_{\ell^2 \to \ell^2}$ was already known earlier, due to the work of Hilbert, Weyl (1908), and especially Schur (1911).
Hilbert’s inequality for general $\ell^p$ spaces

$\|H\|_{\ell^2 \to \ell^2} \leq \pi$ also follows from the well-known Hilbert inequality. More generally, for

$$\ell^p = \{(x_n)_n : \|(x_n)\|_p = \left(\sum_n |x_n|^p\right)^{1/p} < \infty\},$$

Hilbert’s inequality (sharp):

$$\left(\sum_{m=0}^{\infty} \left|\sum_{n=0}^{\infty} \frac{x_n}{m+n+1}\right|^p\right)^{1/p} \leq \frac{\pi}{\sin(\pi/p)} \|x\|_p,$$

where $x = (x_m)_{m=0}^{\infty} \in \ell^p$, $1 < p < \infty$. 
Hilbert’s inequality and further precise estimates imply

$$\|H\|_{\ell^p \to \ell^p} = \frac{\pi}{\sin(\pi/p)} , \quad 1 < p < \infty .$$

When \( p = 2 \), we obtain the already known value \( \pi \).

Note: \( H \) is not bounded on \( \ell^1, \ell^\infty \).

Knowing the norm, a natural further question is: what can we say about the spectrum of \( H \)?

Recall that the \textit{spectrum} of a bounded operator \( T \) acting on a Banach space \( X \) is the set of all \( \lambda \in \mathbb{C} \) such that \( T - \lambda I \) does not have a bounded inverse on \( X \).
**Theorem** (W. Magnus, 1950): The spectrum of $H : \ell^2 \to \ell^2$ is $\sigma(H) = [0, \pi]$. Moreover, it is purely continuous.

In other words:
- the operator $H - \lambda I$ is injective with dense range (not all of $\ell^2$) for each $\lambda \in [0, \pi]$;
- $H - \lambda I$ is invertible for every other complex value of $\lambda$.

In particular, $H$ has no eigenvalues and its spectral radius:

$$r(H) = \lim_{n \to \infty} \|H^n\|^{1/n} = \sup\{ |\lambda| : \lambda \in \mathbb{C} \}$$

is actually $= \|H\| = \pi$. 
**Definition.** A complex number \( \lambda \) is called a *latent eigenvalue* (or *latent root*) of \( H \) if there exists a sequence \( x = (x_n)_n \) that satisfies

\[
\lim_{N \to \infty} \sum_{n=0}^{N} \frac{x_n}{m+n+1} = \lambda x_m.
\]

- O. Taussky-Todd (1949) posed the question as to whether \( \pi \) is a latent eigenvalue of \( H \).
- M. Rosenblum (1958) showed that every \( \lambda \) with \( \text{Re} \lambda > 0 \) is a latent root of \( H \).
- C.K. Hill (1960): extension for generalized Hilbert matrix with \( m + n + \lambda \) instead of \( m + n + 1 \), with \( \lambda \in \mathbb{R} \setminus \{0, -1, -2, \ldots\} \).
$H^p$ spaces

$1 \leq p < \infty$.

Hardy space $H^p$: $f$ analytic in the unit disk $\mathbb{D}$ and

$$
\|f\|_{H^p} = \lim_{r \to 1} \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty.
$$

Special case $p = 2$: if $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ then

$$
\|f\|_{H^2}^2 = \|\hat{f}(n)\|_{\ell^2}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2.
$$

Other $H^p$ spaces are also related to $\ell^p$ but there is no exact correspondence between the norm and the Taylor coefficients.
Is it meaningful to study $H$ as an operator on Hardy spaces? For $p = 2$, obviously yes, since $H^2 \equiv \ell^2$. Just identify a given function $f \in H^2$ with its sequence of Taylor coefficients, $(\hat{f}(n))_{n=0}^{\infty}$, defining $H$ as an operator on $H^2$ by

$$Hf(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} \right) z^n,$$

Magnus did this and observed the key representation formula:

$$Hf(z) = \int_0^1 \frac{f(r)}{1 - rz} dr, \quad z \in \mathbb{D}.$$
Magnus used iteration of integral equations related to the spectrum (such as $Hg = \lambda g$ and similar) to prove his theorem.

E. Diamantopoulos and A. Siskakis (2000) developed Magnus’ idea further, by considering $H$ as an operator acting on analytic functions in $\mathbb{D}$ defined by

$$Hf(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} \right) z^n$$

They studied its action on Hardy spaces.
**Theorem** (Diamantopoulos and Siskakis, 2000). The Hilbert matrix $H$ is bounded on $H^p$ if and only if $1 < p < \infty$. Also, whenever $p \geq 2$, we have

$$\|H\|_{H^p \to H^p} \leq \frac{\pi}{\sin(\pi/p)}.$$ 

- $1 < p < 2$: less precise estimates.
- They used the same integral formula discovered by Magnus but integrated over circular arcs instead of along a radius.
- Representation of $H$ as an integral mean of certain weighted composition operators:

$$Hf(z) = \int_0^1 \frac{1}{1 - (1 - t)z} f\left(\frac{t}{1 - (1 - t)z}\right) dt, \quad z \in \mathbb{D}.$$
Nehari-type estimates for Hankel operators

For \( g \in L^\infty(0, 2\pi) \) and its Fourier coefficients \( \hat{g}(m), m \in \mathbb{Z} \), define the associated Hankel operator \( H_g \) (acting on power series \( f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \) in \( \mathbb{D} \)) by

\[
H_g f(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \hat{g}(n+k) \hat{f}(k) \right) z^n.
\]

**Theorem** (M. Dostanić, M. Jevtić, V., 2008). Let \( 1 < p < \infty \) and \( g \in L^\infty(0, 2\pi) \). Then \( H_g \) is a bounded operator on \( H^p \) and

\[
\|H_g\|_{H^p \to H^p} \leq \frac{\|g\|_{\infty}}{\sin(\pi/p)}.
\]
**Theorem** (Dostanić, Jevtić, V., 2008). For $H_g = H$, the Hilbert matrix (with $g$ picked as before):

$$
\|H\|_{H^p \to H^p} = \frac{\pi}{\sin(\pi/p)}.
$$

- Key point: $H$ can be factored as the Riesz projection (whose norm is known to be $= 1/(\sin(\pi/p))$ by the results of Gokhberg and Krupnik and Hollenbeck and Verbitsky), the operator of multiplication by $g$ (norm $\pi$), and the conjugation (flip) operator (of norm one). This yields an upper bound for the norm.
- Further precise work using Hardy spaces techniques shows that the bound is optimal.
Bergman spaces $A^p$

$$dA(z) = \pi^{-1} \, dx \, dy = \pi^{-1} \, r \, dr \, d\theta.$$ Bergman space: $A^p = L^p(\mathbb{D}, dA) \cap H(\mathbb{D})$: 

$$\|f\|_{A^p} = \left( \int_{\mathbb{D}} |f(z)|^p \, dA(z) \right)^{1/p} = \left( \int_0^1 2r \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right) \, dr \right)^{1/p}.$$ 

Clearly, $H^p \subset A^p$. Actually, $H^p \subset A^{2p}$ (Hardy-Littlewood: more general result; Carleman: sharp inequality).
The Hilbert matrix acting on $A^p$ spaces

**Theorem** (E. Diamantopoulos, 2004). $H$ is bounded on $A^p$ if and only if $2 < p < \infty$. When $4 \leq p < \infty$, we have

$$\|H\|_{A^p \rightarrow A^p} \leq \frac{\pi}{\sin(2\pi/p)}.$$  

His proof used in a clever way a sharp pointwise estimate for $A^p$ functions combined with the techniques developed with Siskakis.

He also obtained some less precise estimate in the case $2 < p < 4$.  

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Spectrum of the Hilbert matrix
Theorem (Dostanić, Jevtić, V., 2008). For $2 < p < \infty$:

$$\|H\|_{A^p \to A^p} \geq \frac{\pi}{\sin(2\pi/p)}.$$  

In particular, $\|H\|_{A^p \to A^p} = \frac{\pi}{\sin(2\pi/p)}$, whenever $4 \leq p < \infty$.

Proof: partly relies on Diamantopoulos’ result. Key point: lower estimate, obtained by considering certain functions that do not belong to the space $A^p$ but “come close”, the latent eigenvectors corresponding to the latent eigenvalue

$$\frac{\pi}{\sin(2\pi/p)}.$$  

Could not get precise value of the norm for $2 < p < 4$.  


**Theorem** (V. Božin, B. Karapetrović, 2018). The same formula also holds for $2 < p < 4$.

Their proof uses a new way of handling the monotonicity of integral means on circles and a clever use of inequalities for the Beta function and of other special functions (Gamma, Hypergeometric).

Generalizations of the above norm computations or estimates to weighted Bergman spaces:
Other recent generalizations


To mixed-norm spaces and Besov spaces: M. Jevtić, B. Karapetrović (2017).

Generalized forms of Hilbert matrix on various spaces of analytic functions (period 2010-2021):
B. Łanucha, M. Nowak, M. Pavlović; C. Chatzifountas, P. Galanopoulos, D. Girela, N. Merchán; J. Rättyä, J.A. Peláez, E. de la Rosa; T, Kalivoda, P. Štoviček, etc.
Further study of eigenvalues

• A. Aleman, A. Montes-Rodríguez, A. Saraooleanu (2012) considered the matrix $H_\lambda$ with entries $a_{n,k} = (n + k + \lambda)^{-1}$, with $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.

• They showed that the operator $H_\lambda$ induced by this matrix preserves both the Hardy spaces $H^p$, $1 < p < \infty$, and also the Korenblum (growth) spaces $A^{-\alpha}$, $\alpha > 0$, consisting of functions $f$ analytic in the disk such that

$$\sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f(z)| < \infty.$$
Aleman, Montes-Rodríguez, and Sarafoleanu were inspired by the work of Rosenblum, and Hill.

They also found some differential operators that “almost commutes” with $H_\lambda$: $(Dh - HD)f = f(0)$, for example:

$$Df(z) = (1 - z^2)f'(z) - zf(z).$$

They used this relation to find the eigenfunctions and eigenvalues and obtain interesting relationships with special functions.

Again, the hypergeometric functions play a prominent role, like in so many problems in analysis.
Most recent study of the spectrum: Silbermann

One can go far beyond the norm or eigenvalues, even in the settings like $\ell^p$, $H^p$, or weighted $A^p$. However, this has been established only recently and not all the results have been published yet.

B. Silbermann (2021) managed to determine the spectrum of $H$ on both $H^p$ and $\ell^p$ spaces.

- One important point: the spectrum coincides with a range of certain function that one can visualize.
• Tools used by Silbermann: Banach algebras and Gelfand transform, Fredholm theory of Toeplitz and Hankel operators, some old results by Gokhberg and Krupnik, Hausdorff-Young theorem, special functions, results by Aleman et al., and a variety of other techniques.

• A. Aleman, A. Siskakis, D.V. (paper still in preparation): Results in the same vein announced earlier, using a very different approach for general families of spaces.

• Initially, we also noticed the above phenomenon and used the Fourier transform to compute the spectrum of other related operators (for example, of a Hilbert matrix-type operator acting on the Hardy space of the half-plane) but this did not really lead to too many new results.
• Namely, work by Kalivoda and Štoviček from 2016 already contained very similar results for Carleman-type operators. Also that of D.R. Yafaev (2014-15), formulated in the language of Mathematical Physics (requires translations of concepts). Interesting older work: E. Fabes, M. Jodeit, J. Lewis on the spectra of a Hardy kernel (1976).

• This has led us to change the approach, going back to the spaces of the disk but enlarging the family of spaces considered (thus leading to a more general approach). We sketch briefly our approach in [ASV].
New theory of conformal invariance (with $\alpha > 0$, different from the classical conformal invariance with $\alpha = 0$) was proposed in 2019 by A. Aleman and A. Mas (published in 2021).

Let $\varphi_a(z) = \frac{a+z}{1+az}$ be a typical disk automorphism.

A Banach space $X$ of analytic functions in $\mathbb{D}$ is \textit{conformally invariant of index $\alpha$} if the operators $W^\alpha_a$ defined by

$$W^\alpha_a f = (\varphi'_a)^\alpha(f \circ \varphi_a)$$

are uniformly bounded and bounded below, independently of $a$. 
If such a value $\alpha = \alpha(X)$ exist, then it turns out to be unique. Examples:

$$\alpha(H^p) = \frac{1}{p}, \quad \alpha(A^p) = \frac{2}{p}, \quad \alpha(A^{-\gamma}) = \gamma.$$  

Many other spaces also have this property. The sequence space $\ell^p$, interpreted as a space of analytic functions, does not (G. Halász, 1967).

It turns out that each conformally invariant space of index $\alpha$ contains a certain Dirichlet-type space and is contained in a small Korenblum type space (whose exponents depend on $\alpha$): Aleman, Mas.
Apparently, $H$ and $W_\alpha^\alpha$ are completely unrelated. Nonetheless, it turns out that the number $\alpha(X)$ completely determines the spectrum of $H$ when it is bounded on the space $X$.

In the paper in preparation [ASV], this approach allows us to consider Banach spaces $X$ of analytic functions that satisfy only a handful of properties:

1. $X$ is continuously embedded into $\mathcal{H}(\mathbb{D})$,
2. Polynomials are dense in $X$ and also in its Cauchy dual $X'$,
3. There exists $m$ s.t. $C^m(\overline{\mathbb{D}} \cap \mathcal{H}(\mathbb{D}) \subset Mult(X)$,
4. The operator $U$ defined by $Uf(z) = f(-z)$ is bounded on $X$,
5. $X$ is conformally invariant of index $\alpha = \alpha(X)$.
Let, as before
\[ W_{a^{1/2}} f = (\varphi'_a)^{1/2}(f \circ \varphi_a). \]

Then it turns out that the operators \( \{ W_{a^{1/2}} : a \in (-1, 1) \} \) form a strongly continuous group on \( \mathcal{B}(X) \), the space of bounded linear operators on \( X \), for any of the spaces \( X \) considered.

The \textit{infinitesimal generator} \( D \) of this group, defined for all \( f \in X \) for which the limit
\[ Df = \lim_{a \to 0} \frac{1}{a}(W_{a} - I)f \]
exists, is easily computed: \( Df(z) = (1 - z^2)f'(z) - zf(z) \), an operator seen before (AMS, 2012). Moreover,
\[ \sigma(D) = S_\alpha = \{ \lambda \in \mathbb{C} : |\Re \lambda| \leq |1 - 2\alpha| \}. \]
One important idea: obtain the spectral information on the \textit{companion operator} given by

$$Kf(z) = \int_{-1}^{1} \frac{f(r)}{1 - rz} \, dr, \quad z \in \mathbb{D}.$$ 

Its matrix is similar to the one of $H$ but with zeros instead of terms with even integers on the corresponding diagonals.

Both operators $H$ and $K$ turn out to be bounded on all spaces that satisfy the required axioms.

It is not difficult to see (integration) that

$$Kf(z) = \int_{-1}^{1} W_{a}^{1/2} f(z) \frac{da}{\sqrt{1 - a^2}}.$$
A version of the spectral mapping theorem gives

\[ \sigma(K) = \{0\} \cup \left\{ \int_{-1}^{1} \left(\frac{1+a}{1-a}\right)^{\lambda/2} \frac{da}{\sqrt{1-a^2}} : \lambda \in S_\alpha \right\}, \]

where

\[ S_\alpha = \{ \lambda \in \mathbb{C} : |\text{Re} \, \lambda| \leq |1 - 2\alpha| \}. \]

A computation yields

\[ \sigma(K) = \{0\} \cup \left\{ \frac{\pi}{\cos \frac{\pi \lambda}{2}} : \lambda \in S_\alpha \right\}, \]

A priori it does not look easy to obtain the spectrum of \( H \) from that of \( K \). But this can be done.
It turns out that $H$ shares many spectral properties with $K$, e.g.,

$$\sigma(H) = \sigma(K) = \{0\} \cup \left\{ \frac{\pi}{\cos \frac{\pi \lambda}{2}} : \lambda \in S_\alpha \right\},$$

If $\alpha = 1/2$, the spectrum of $H$ (or $K$) is $[0; \pi]$, as in the work by Magnus.

If $\alpha \neq 1/2$, the spectrum of $H$ (or $K$) has non-empty interior. Its boundary is a Jordan curve symmetric w.r.t. the real axis which passes through 0 and whose interior contains the interval $(0, \pi]$ (earlier picture, like Silbermann). The spectral radius of both operators is

$$r(H) = \lim_{n \to \infty} \|H^n\|^{1/n} = \frac{\pi}{\cos \left( \pi|\alpha - \frac{1}{2}| \right)}.$$
It has been verified in some recent works (M. Lindström et al.) that for $H^p$ and some weighted Bergman spaces, this value coincides with the norm of $H$.

In our work, we also make use of the fact that certain operator that relates $H$ and $K$ is compact, analysis of parts of the spectrum, Gauss’ hypergeometric functions of a special type, the Gelfand transform of the operator $K$ in a certain maximal commutative Banach algebra that contains our group, etc. Need some further arguments to cover $\ell^p$ spaces (this can be done).
THANK YOU VERY MUCH!