On the spectrum of the Hilbert matrix

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Hankel operators on ℓ^2

Space of square-summable (complex) sequences:

$$\ell^2 = \{(x_n)_n : \ \|(x_n)\|_2 = \left(\sum_n |x_n|^2\right)^{1/2} < \infty\}.$$

Definition. A *Hankel operator* on the space ℓ^2 is an operator defined by a matrix whose entries $a_{m,n}$ depend only on the sum of the coordinates: $a_{m,n} = c_{m+n}$ for some sequence $(c_k)_{k=0}^{\infty}$:

$$\begin{bmatrix} c_0 & c_1 & c_2 & \dots \\ c_1 & c_2 & c_3 & \dots \\ c_2 & c_3 & c_4 & \dots \\ \dots & & & \dots \end{bmatrix}$$

Nehari's ℓ^2-theorem

Fourier coefficients of $g \in L^\infty(0, 2\pi)$:

$$\hat{g}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} g(t)dt \,, \qquad n \in \mathbb{Z} \,.$$

Theorem (Nehari, 1957). Let H_a be a Hankel operator defined by the matrix with entries $a_{m,n} = c_{m+n}$. Then H_a is bounded on ℓ^2 if and only if there exists a function $g \in L^\infty(0, 2\pi)$ such that $c_k = \hat{g}(k)$, $k \geq 0$.

$$\|H_a\|_{\ell^2 \to \ell^2} = \inf \|g\|_{L^\infty(\mathbb{T})} \,,$$

where the infimum is taken over all functions g as above.

The Hilbert matrix H

Infinite matrix with entries $a_{m,n} = (m+n+1)^{-1}$, $m, n \geq 0$:

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \cdots & & & \cdots \end{bmatrix}$$

Action on an infinite sequence $x = (x_n)_{n=0}^{\infty}$: the n-th coordinate is

$$(Hx)_n = \left(\sum_{n=0}^{\infty} \frac{x_n}{m+n+1} \right)_{m=0}^{\infty} .$$

Basic facts about the Hilbert matrix

- It was introduced by Hilbert in 1894 in the study of orthogonal polynomials.
- It arises naturally as the Gram matrix of the system $(x^n)_{n=0}^{\infty}$ in $L^2[0,1]$: $\langle x^m, x^n \rangle = (m+n+1)^{-1}$.
- It is a prototype of a Hankel operator on ℓ^2: $a_{m,n} = (m+n+1)^{-1}$.
- It is a bounded and self-adjoint operator on ℓ^2.
- Its norm on ℓ^2 can be computed using Nehari's theorem or Hilbert's inequality.

Norm of H on ℓ^2

For the Hilbert matrix H, the choice

$$g(t) = ie^{-it}(\pi - t)\,, \quad 0 \le t < 2\pi\,,$$

in Nehari's theorem yields

$$\|H\|_{\ell^2 \to \ell^2} \le \|g\|_{L^\infty(\mathbb{T})} = \pi\,.$$

It can be shown that the norm is exactly π.

Although the general Nehari theorem was discovered much later, the value of $\|H\|_{\ell^2 \to \ell^2}$ was already known earlier, due to the work of Hilbert, Weyl (1908), and especially Schur (1911).

Hilbert's inequality for general ℓ^p spaces

$\|H\|_{\ell^2 \to \ell^2} \leq \pi$ also follows from the well-known Hilbert inequality. More generally, for

$$\ell^p = \{(x_n)_n : \|(x_n)\|_p = \left(\sum_n |x_n|^p \right)^{1/p} < \infty\},$$

Hilbert's inequality (sharp):

$$\left(\sum_{m=0}^{\infty} \left| \sum_{n=0}^{\infty} \frac{x_n}{m+n+1} \right|^p \right)^{1/p} \leq \frac{\pi}{\sin(\pi/p)} \|x\|_p , \qquad (1)$$

where $x = (x_m)_{m=0}^{\infty} \in \ell^p$, $1 < p < \infty$.

Norm of H as an operator on ℓ^p

Hilbert's inequality and further precise estimates imply

$$\|H\|_{\ell^p \to \ell^p} = \frac{\pi}{\sin(\pi/p)}\,, \quad 1 < p < \infty\,.$$

When $p = 2$, we obtain the already known value π.

Note: H is not bounded on ℓ^1, ℓ^∞.

Knowing the norm, a natural further question is: what can we say about the spectrum of H?

Recall that the *spectrum* of a bounded operator T acting on a Banach space X is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ does not have a bounded inverse on X.

Spectrum of H acting on ℓ^2

Theorem (W. Magnus, 1950): The spectrum of $H: \ \ell^2 \to \ell^2$ is $\sigma(H) = [0, \pi]$.

Moreover, it is purely continuous.

In other words:
- the operator $H - \lambda I$ is injective with dense range (not all of ℓ^2) for each $\lambda \in [0, \pi]$;
- $H - \lambda I$ is invertible for every other complex value of λ.

In particular, H has no eigenvalues and its spectral radius:

$$r(H) = \lim_{n \to \infty} \|H^n\|^{1/n} = \sup\{|\lambda| \ : \ \lambda \in \mathbb{C}\}$$

is actually $= \|H\| = \pi$.

Latent eigenvalues

Definition. A complex number λ is called a *latent eigenvalue* (or *latent root*) of H if there exists a sequence $x = (x_n)_n$ that satisfies

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{x_n}{m+n+1} = \lambda x_m \,.$$

- O. Taussky-Todd (1949) posed the question as to whether π is a latent eigenvalue of H.
- T. Kato (1957) gave an affirmative answer.
- M. Rosenblum (1958) showed that every λ with $\operatorname{Re}\lambda > 0$ is a latent root of H.
- C.K. Hill (1960): extension for generalized Hilbert matrix with $m+n+\lambda$ instead of $m+n+1$, with $\lambda \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$.

H^p spaces

$1 \le p < \infty$.
Hardy space H^p: f analytic in the unit disk $\mathbb{D}$ and

$$\|f\|_{H^p} = \lim_{r\to 1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty \, .$$

Special case $p = 2$: if $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)\, z^n$ then

$$\|f\|_{H^2}^2 = \|(\hat{f}(n))\|_{\ell^2}^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \, .$$

Other H^p spaces are also related to ℓ^p but there is no exact correspondence between the norm and the Taylor coefficients.

The Hilbert matrix operator on H^p spaces

Is it meaningful to study H as an operator on Hardy spaces? For $p = 2$, obviously yes, since $H^2 \equiv \ell^2$. Just identify a given function $f \in H^2$ with its sequence of Taylor coefficients, $(\hat{f}(n))_{n=0}^{\infty}$, defining H as an operator on H^2 by

$$Hf(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} \right) z^n \,,$$

Magnus did this and observed the key representation formula:

$$Hf(z) = \int_0^1 \frac{f(r)}{1-rz} dr \,, \quad z \in \mathbb{D} \,.$$

Magnus used iteration of integral equations related to the spectrum (such as $Hg = \lambda g$ and similar) to prove his theorem.

E. Diamantopoulos and A. Siskakis (2000) developed Magnus' idea further, by considering H as an operator acting on analytic functions in $\mathbb{D}$ defined by

$$Hf(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} \right) z^n$$

They studied its action on Hardy spaces.

Theorem (Diamantopoulos and Siskakis, 2000). The Hilbert matrix H is bounded on H^p if and only if $1 < p < \infty$. Also, whenever $p \geq 2$, we have

$$\|H\|_{H^p \to H^p} \leq \frac{\pi}{\sin(\pi/p)}\,.$$

• $1 < p < 2$: less precise estimates.

• They used the same integral formula discovered by Magnus but integrated over circular arcs instead of along a radius.

• Representation of H as an integral mean of certain weighted composition operators:

$$Hf(z) = \int_0^1 \frac{1}{1-(1-t)z} f\left(\frac{t}{1-(1-t)z}\right) dt\,, \quad z \in \mathbb{D}\,.$$

Nehari-type estimates for Hankel operators

For $g \in L^\infty(0, 2\pi)$ and its Fourier coefficients $\hat{g}(m)$, $m \in \mathbb{Z}$, define the associated Hankel operator H_g (acting on power series $f(z) = \sum_{n=0}^\infty \hat{f}(n) z^n$ in $\mathbb{D}$) by

$$H_g f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \hat{g}(n+k) \hat{f}(k) \right) z^n .$$

Theorem (M. Dostanić, M. Jevtić, V., 2008). Let $1 < p < \infty$ and $g \in L^\infty(0, 2\pi)$. Then H_g is a bounded operator on H^p and

$$\|H_g\|_{H^p \to H^p} \le \frac{\|g\|_\infty}{\sin(\pi/p)} .$$

Exact value of the norm on H^p

Theorem (Dostanić, Jevtić, V., 2008). For $H_g = H$, the Hilbert matrix (with g picked as before):

$$\|H\|_{H^p \to H^p} = \frac{\pi}{\sin(\pi/p)} .$$

• Key point: H can be factored as the Riesz projection (whose norm is known to be $= 1/(\sin(\pi/p))$ by the results of Gokhberg and Krupnik and Hollenbeck and Verbitsky), the operator of multiplication by g (norm π), and the conjugation (flip) operator (of norm one). This yields an upper bound for the norm.

• Further precise work using Hardy spaces techniques shows that the bound is optimal.

Bergman spaces A^p

$dA(z) = \pi^{-1} dx\, dy = \pi^{-1} r\, dr\, d\theta$. Bergman space:
$A^p = L^p(\mathbb{D}, dA) \cap \mathcal{H}(\mathbb{D})$:

$$\begin{aligned} \|f\|_{A^p} &= \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z) \right)^{1/p} \\ &= \left(\int_0^1 2r \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right) dr \right)^{1/p} . \end{aligned}$$

Clearly, $H^p \subset A^p$. Actually, $H^p \subset A^{2p}$ (Hardy-Littlewood: more general result; Carleman: sharp inequality).

The Hilbert matrix acting on A^p spaces

Theorem (E. Diamantopoulos, 2004). H is bounded on A^p if and only if $2 < p < \infty$. When $4 \le p < \infty$, we have

$$\|H\|_{A^p \to A^p} \le \frac{\pi}{\sin(2\pi/p)} .$$

His proof used in a clever way a sharp pointwise estimate for A^p functions combined with the techniques developed with Siskakis.
He also obtained some less precise estimate in the case $2 < p < 4$.

Theorem (Dostanić, Jevtić, V., 2008). For $2 < p < \infty$:

$$\|H\|_{A^p \to A^p} \geq \frac{\pi}{\sin(2\pi/p)} .$$

In particular, $\|H\|_{A^p \to A^p} = \frac{\pi}{\sin(2\pi/p)}$, whenever $4 \leq p < \infty$.

Proof: partly relies on Diamantopoulos' result. Key point: lower estimate, obtained by considering certain functions that do not belong to the space A^p but "come close", the latent eigenvectors corresponding to the latent eigenvalue

$$\frac{\pi}{\sin(2\pi/p)} .$$

Could not get precise value of the norm for $2 < p < 4$.

Theorem (V. Božin, B. Karapetrović, 2018). The same formula also holds for $2 < p < 4$.

Their proof uses a new way of handling the monotonicity of integral means on circles and a clever use of inequalities for the Beta function and of other special functions (Gamma, Hypergeometric).

Generalizations of the above norm computations or estimates to weighted Bergman spaces:
B. Karapetrović (2018); M. Lindström, S. Miihkinen, N. Wikman (2021), D. Bralović, B. Karapetrović (preprint).

Other recent generalizations

Generalizations to Korenblum-type (growth spaces): M. Lindström, S. Miihkinen, N. Wikman (2019).

To mixed-norm spaces and Besov spaces: M. Jevtíc, B. Karapetrović (2017).

Generalized forms of Hilbert matrix on various spaces of analytic functions (period 2010-2021):
B. Łanucha, M. Nowak, M. Pavlović; C. Chatzifountas, P. Galanopoulos, D. Girela, N. Merchán; J. Rättyä, J.A. Peláez, E. de la Rosa; T, Kalivoda, P. Štoviček, etc.

Further study of eigenvalues

• A. Aleman, A. Montes-Rodríguez, A. Sarafoleanu (2012) considered the matrix H_λ with entries $a_{n,k} = (n+k+\lambda)^{-1}$, with $\lambda \in \mathbb{C} \setminus \mathbb{Z}$.

• They showed that the operator H_λ induced by this matrix preserves both the Hardy spaces H^p, $1 < p < \infty$, and also the Korenblum (growth) spaces $A^{-\alpha}$, $\alpha > 0$, consisting of functions f analytic in the disk such that

$$\sup_{z \in \mathbb{D}} (1-|z|)^\alpha |f(z)| < \infty \,.$$

• Aleman, Montes-Rodríguez, and Sarafoleanu were inspired by the work of Rosenblum, and Hill.

• They also found some differential operators that “almost commutes” with H_λ: $(Dh - HD)f = f(0)$, for example:

$$Df(z) = (1 - z^2)f'(z) - zf(z)\,.$$

They used this relation to find the eigenfunctions and eigenvalues and obtain interesting relationships with special functions.

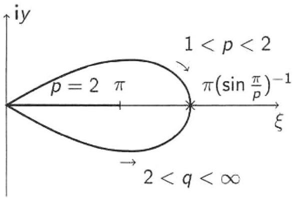
• Again, the hypergeometric functions play a prominent role, like in so many problems in analysis.

Most recent study of the spectrum: Silbermann

One can go far beyond the norm or eigenvalues, even in the settings like ℓ^p, H^p, or weighted A^p. However, this has been established only recently and not all the results have been published yet.

B. Silbermann (2021) managed to determine the spectrum of H on both H^p and ℓ^p spaces.

• One important point: the spectrum coincides with a range of certain function that one can visualize.



• Tools used by Silbermann: Banach algebras and Gelfand transform, Fredholm theory of Toeplitz and Hankel operators, some old results by Gokhberg and Krupnik, Hausdorff-Young theorem, special functions, results by Aleman *et al.*, and a variety of other techniques.

• A. Aleman, A. Siskakis, D.V. (paper still in preparation): Results in the same vein announced earlier, using a very different approach for general families of spaces.

• Initially, we also noticed the above phenomenon and used the Fourier transform to compute the spectrum of other related operators (for example, of a Hilbert matrix-type operator acting on the Hardy space of the half-plane) but this did not really lead to too many new results.

- Namely, work by Kalivoda and Štoviček from 2016 already contained very similar results for Carleman-type operators. Also that of D.R. Yafaev (2014-15), formulated in the language of Mathematical Physics (requires translations of concepts). Interesting older work: E. Fabes, M. Jodeit, J. Lewis on the spectra of a Hardy kernel (1976).
- This has led us to change the approach, going back to the spaces of the disk but enlarging the family of spaces considered (thus leading to a more general approach).

We sketch briefly our approach in [ASV].

A new concept of conformal invariance

New theory of conformal invariance (with $\alpha > 0$, different from the classical conformal invariance with $\alpha = 0$) was proposed in 2019 by A. Aleman and A. Mas (published in 2021).

Let $\varphi_a(z) = \frac{a+z}{1+\bar{a}z}$ be a typical disk automorphism.

A Banach space X of analytic functions in $\mathbb{D}$ is *conformally invariant of index* α if the operators W_a^α defined by

$$W_a^\alpha f = (\varphi_a')^\alpha (f \circ \varphi_a)$$

are uniformly bounded and bounded below, independently of a.

If such a value $\alpha = \alpha(X)$ exist, then it turns out to be unique.
Examples:

$$\alpha(H^p) = \frac{1}{p}\,, \qquad \alpha(A^p) = \frac{2}{p}\,, \qquad \alpha(A^{-\gamma}) = \gamma\,.$$

Many other spaces also have this property. The sequence space ℓ^p, interpreted as a space of analytic functions, does not (G. Halász, 1967).

It turns out that each conformally invariant space of index α contains a certain Dirichlet-type space and is contained in a small Korenblum type space (whose exponents depend on α): Aleman, Mas.

Most recent study of the spectrum: our approach

Apparently, H and W_a^α are completely unrelated. Nonetheless, it turns out that the number $\alpha(X)$ completely determines the spectrum of H when it is bounded on the space X.

In the paper in preparation [ASV], this approach allows us to consider Banach spaces X of analytic functions that satisfy only a handful of properties:

(1) X is continuously embedded into $\mathcal{H}(\mathbb{D})$,
(2) Polynomials are dense in X and also in its Cauchy dual X',
(3) There exists m s.t. $C^m(\overline{\mathbb{D}}) \cap \mathcal{H}(\mathbb{D}) \subset Mult(X)$,
(4) The operator U defined by $Uf(z) = f(-z)$ is bounded on X,
(5) X is conformally invariant of index $\alpha = \alpha(X)$.

Let, as before

$$W_a^{1/2} f = (\varphi_a')^{1/2} (f \circ \varphi_a) \,.$$

Then it turns out that the operators $\{W_a^{1/2} \,:\, a \in (-1,1)\}$ form a strongly continuous group on $\mathcal{B}(X)$, the space of bounded linear operators on X, for any of the spaces X considered.

The *infinitesimal generator* D of this group, defined for all $f \in X$ for which the limit

$$Df = \lim_{a \to 0} \frac{1}{a} (W_a - I) f$$

exists, is easily computed: $Df(z) = (1 - z^2) f'(z) - z f(z)$, an operator seen before (AMS, 2012). Moreover,

$$\sigma(D) = S_\alpha = \{\lambda \in \mathbb{C} : \; |\mathrm{Re}\, \lambda| \le |1 - 2\alpha|\} \,.$$

One important idea: obtain the spectral information on the *companion operator* given by

$$Kf(z) = \int_{-1}^{1} \frac{f(r)}{1 - rz} dr\,, \quad z \in \mathbb{D}\,.$$

Its matrix is similar to the one of H but with zeros instead of terms with even integers on the corresponding diagonals.

Both operators H and K turn out to be bounded on all spaces that satisfy the required axioms.

It is not difficult to see (integration) that

$$Kf(z) = \int_{-1}^{1} W_a^{1/2} f(z) \frac{da}{\sqrt{1 - a^2}}\,.$$

A version of the spectral mapping theorem gives

$$\sigma(K) = \{0\} \cup \left\{ \int_{-1}^{1} \left(\frac{1+a}{1-a} \right)^{\lambda/2} \frac{da}{\sqrt{1-a^2}} \; : \; \lambda \in S_\alpha \right\},$$

where

$$S_\alpha = \{\lambda \in \mathbb{C} : \; |\mathrm{Re}\ \lambda| \le |1-2\alpha|\}.$$

A computation yields

$$\sigma(K) = \{0\} \cup \left\{ \frac{\pi}{\cos \frac{\pi\lambda}{2}} \; : \; \lambda \in S_\alpha \right\},$$

A priori it does not look easy to obtain the spectrum of H from that of K. But this can be done.

It turns out that H shares many spectral properties with K, *e.g.*,

$$\sigma(H) = \sigma(K) = \{0\} \cup \left\{ \frac{\pi}{\cos \frac{\pi\lambda}{2}} \,:\, \lambda \in S_\alpha \right\},$$

If $\alpha = 1/2$, the spectrum of H (or K) is $[0;\pi]$, as in the work by Magnus.

If $\alpha \neq 1/2$, the spectrum of H (or K) has non-empty interior. Its boundary is a Jordan curve symmetric w.r.t. the real axis which passes through 0 and whose interior contains the interval $(0,\pi]$ (earlier picture, like Silbermann). The spectral radius of both operators is

$$r(H) = \lim_{n\to\infty} \|H^n\|^{1/n} = \frac{\pi}{\cos\left(\pi|\alpha - \frac{1}{2}|\right)}.$$

It has been verified in some recent works (M. Lindström *et al.*) that for H^p and some weighted Bergman spaces, this value coincides with the norm of H.

In our work, we also make use of the fact that certain operator that relates H and K is compact, analysis of parts of the spectrum, Gauss' hypergeometric functions of a special type, the Gelfand transform of the operator K in a certain maximal commutative Banach algebra that contains our group, etc.

Need some further arguments to cover ℓ^p spaces (this can be done).

THANK YOU VERY MUCH!